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On the Classical Derivation of Electrodynamic Equations from the Stationary Action

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Abstract:- The Principle of Stationary Action is extremely useful and plays a central role in deriving physics equations. One of the demanding aspects of this topic, difficult to explain, is how it connects with electrodynamic equations. This paper presents a simple derivation of classical electrodynamic equations based on Stationary action principle in which the Lagrangian formalism of a nonrelativistic mechanical system is extended to obtain the relativistic Lagrangian equation of a free particle in an external field. Lorentz invariance and appropriate action integrals for the moving particles in static fields, moving fields, and matter-field interaction are constructed to obtain the equations of motion of charged particles. Inhomogeneous Maxwell's equations of the electromagnetic field are obtained using electromagnetic field tensor with six independent components of electric field (E) and magnetic field (B) in matrix form via the electromagnetic Lagrangian density. It is also considered that the homogeneous part and the Bianchi identity are derived by introducing a dual field tensor. The continuity equation of motion is presented by introducing electromagnetic 4-divergence.

Keywords:- *Electrodynamics, Maxwell's Equations, Bianchi Identity, Lorentz Force And Least Action Principle.*

I. INTRODUCTION

Classical electrodynamics, usually taught at the graduate level, is a key branch of physics theory that explains the electromagnetic forces between the electric charges and currents. Most importantly, the classical prediction from a planetry model that atom would be unstable. Hence, it is encouraged to understand the electrodynamic phenomenon and its fundamental equations from a unique point of view. A great tool for the starting point is the principle of least action or uniquely the principle of stationary action. This principle is extremely useful and the central part of all physics equations. A clear picture of this principle has been given in classical mechanics class. To shortly dive into it, to formulate the classical electrodynamic equations from the point of view of the stationary action. The formulation needs to be based on the principle that considers the entire motion of the system of particles from a configuration at time t_1 to another configuration at time t_2. This principle is called ***the least action principle*** which states that for a nonrelativistic mechanical system, the action integral.

$S = \int_{t_1}^{t_2} L[q_i(t), \dot{q}_i(t), t]dt\,, i = 1,2 \dots n$ (1) is an extremum [1]. The Lagrangian L in (1) is a function of the generalized coordinates $q_i(t)$ and velocities $\dot{q}_i(t)$. By considering the infinitesimal variations of $q_i(t)$ and $\dot{q}_i(t)$

$$\delta S = \delta \int_{t_1}^{t_2} L[q_i(t), \dot{q}_i(t), t]dt = 0 \qquad (2)$$

The corresponding Euler-Lagrange equation of motion becomes [1]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \qquad (3)$$

One of the frustrated aspects of this topic is how it gives rise to electrodynamic equations. So, this paper employs a simple and approachable method to show how the least action principle is applicable to classical electrodynamics. This principle is applied in a simple way to derive the iconic equations of the classical electrodynamics and to create a path way for the derivation of others. This is done by extending the Lagrangian formalism of a nonrelativistic mechanical system to obtain the relativistic Lagrangian equation of a free particle in an external field. Consequently, the construction of a unique Lorentz invariance and appropriate action integrals for the moving particles in static fields, moving fields, and matter-field interaction to obtain the equations of motion of charged particles. To verify the Lorentz force law, the action integral of free relativistic particles in the static background fields and the interaction of the field are considered under the constructed Lorentz invariance condition. By writing Lorentz force in 4 vector form, it is required to introduce electromagnetic field tensor with six independent components of electric field **E** and magnetic field **B** in matrix form. This is used to generate the inhomogeneous Maxwell's equations of the electromagnetic field (not static) through the electromagnetic Lagrangian density. To complete the demonstration of the electrodynamics 4-vector covariance, a dual field tensor is intoduced to obtain the homogeneous part of the Maxwell's equations and the Bianchi identity. By introducing the 4-divergence, the conservation of source charge density is obtained which leads to the continuity equation of motion.

II. RELATIVISTIC LAGRANGIAN FORMULATION

The mechanical Lagrangian formalism can be extended to obtain relativistic Lagrangian equation of motion for a free particle in external fields. The only Lorentz invariance function for such a free relativistic particle is [2-4, 8]

$$dS^2 = dx_\mu dx^\mu \tag{4}$$

Which does not depend on the origin of time and space. The action integral S is constructed in the form

$$S = \int_1^2 \left(dx_\mu dx^\mu\right)^{\frac{1}{2}} \tag{5}$$

Where $dx_\mu dx^\mu = C^2 dt^2 - dx^2$ and $V = \frac{dx}{dt}$ is the velocity of the particle, the action integral becomes

$$S = C\beta \int_{\tau_1}^{\tau_2} dt \left(1 - \frac{V^2}{C^2}\right)^{\frac{1}{2}} \tag{6}$$

where $dt = \gamma d\tau$, τ is the proper time which is invariant and β is the velocity of particles relative to the speed of light C. The condition that the action S be invariant requires the βL also be invariant. Comparing (6) with (1) gives the Lagrangian L for a free particle to be

$$L = C\beta \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \tag{7}$$

From the Taylor series expansion with the condition that $\frac{V}{c} \ll 1$ (nonrelativistic limit)

$$\left(1 - \frac{V^2}{C^2}\right)^{\frac{1}{2}} \approx 1 - \frac{v^2}{2c^2} \tag{8}$$

To obtain β, (8) is substituted into (7) and $L = \frac{1}{2}mv^2$ for nonrelativistic system of particles, the constant $C\beta$ in the first term is ignored since its derivative is zero. Then, β in the second term is obtained to be $-mc$ which is substituted back into (7) to give

$$L_f = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \tag{9}$$

Equation (9) is termed ***the Lagrangian equation for a free particle*** denoted by L_f which depends not on the position of the particle but on the velocity and mass of the particle.

III. LORENTZ FORCE

In electrostatic, electric field, **E** is known to be written in terms of electrostatic scalar potential Φ and vector potential **A** according to the equation $\mathbf{E} = \frac{\partial A}{\partial t} - \nabla\Phi$. The magnetic field is written in terms of vector potential as in $\boldsymbol{B} = \boldsymbol{\nabla x\, A}$. The potentials ($\Phi$ and $\boldsymbol{A}$) have no physical meaning; they are introduced mainly for mathematical simplification. Lorentz force is derived by considering the case of the charge particles moving in static background fields **E** and **B**. To derive the equation of motion of such particles, the Lagrangian equation is written in two ways: the Lagrangian of the particles in motion L_f and that of the interacting field L_{int}. The condition that warrant the action S to be invariant requires that βL_{int} is also Lorentz invariance which is linear in the field, linear in the charge of the particles and linear in the coordinate. For this reason, βL_{int} is described as the product of the 4-vector potential A^μ for $A^\mu \rightarrow \left(\frac{\Phi}{c}, \boldsymbol{A}\right)$. The possible invariant action integral for the interacting field is [3]

$$\begin{aligned} S_{int} &= q\int_{t_1}^{t_2} A^\mu \, dx_\mu = q\int_{t_1}^{t_2}\left[\frac{\Phi}{c}.\mathrm{cdt} - \mathbf{A}\mathrm{dx}\right] \\ &= q\int_{t_1}^{t_2}[\Phi - \mathbf{A}\mathrm{V}]dt \end{aligned} \tag{10}$$

The Lagrangian of the interacting field is

$$L_{int} = q[\Phi - \mathbf{A}\mathrm{V}] \tag{11}$$

The total Lagrangian $L = L_f + L_{int}$

$$L = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + \frac{\mathrm{e}}{\mathrm{c}}\mathbf{Av} - e\Phi \tag{12}$$

By applying the Euler-Lagrange equation from (3)

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \mathbf{v}}\right) = \frac{\partial \mathrm{L}}{\partial \mathbf{r}}$$

The canonical momentum of the particle is

$$\begin{aligned} \mathbf{P} = \frac{\partial \mathrm{L}}{\partial \mathbf{v}} &= \frac{\partial}{\partial \mathbf{v}}\left[-mc^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + \frac{\mathrm{e}}{\mathrm{c}}\mathbf{Av} - e\Phi\right] \\ &= \gamma\mathrm{m}\mathbf{v} + \frac{\mathrm{e}}{\mathrm{c}}\mathbf{A} \end{aligned} \tag{13}$$

where **P** is the conjugate momentum and the $\gamma\mathrm{m}\mathbf{v}$ is regarded as the ordinary kinetic momentum and $\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}$. Taking $\frac{\partial \mathrm{L}}{\partial \mathbf{r}}$ in (12) as the gradient of ($\frac{\mathrm{e}}{\mathrm{c}}\mathbf{Av} - e\Phi$), the

Euler-Lagrange equation of motion then becomes

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\gamma \mathrm{m}\mathbf{v}+\frac{\mathrm{e}}{\mathrm{c}}\mathbf{A}\right)=\frac{\mathrm{e}}{\mathrm{c}}\boldsymbol{\nabla}(\mathbf{A}\mathbf{v})-e\boldsymbol{\nabla}\Phi$$

$$\left(\gamma\frac{\mathrm{d}(\mathrm{m}\mathbf{v})}{\mathrm{dt}}+\frac{\mathrm{e}}{\mathrm{c}}\frac{\mathrm{d}\mathbf{A}}{\mathrm{dt}}\right)=\frac{\mathrm{e}}{\mathrm{c}}(\mathbf{v}.\boldsymbol{\nabla})\mathbf{A}+\frac{\mathrm{e}}{\mathrm{c}}\mathbf{v}\mathrm{x}(\boldsymbol{\nabla}\mathrm{x}\mathbf{A})-e\boldsymbol{\nabla}\Phi$$

Where $\frac{\mathbf{dA}}{\mathrm{dt}}=\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}.\boldsymbol{\nabla})\mathbf{A}$, $\mathbf{E}=\frac{\partial \boldsymbol{A}}{\partial t}-\boldsymbol{\nabla}\Phi$, $\mathbf{B}=\boldsymbol{\nabla}\mathrm{x}\mathbf{A}$
The ***Lorentz force law*** becomes

$$F=\frac{\mathrm{d}\mathbf{P}}{\mathrm{dt}}=\mathrm{e}\mathbf{E}+\frac{\mathrm{e}}{\mathrm{c}}(\mathbf{v}\ \mathbf{x}\ \mathbf{B}) \tag{14}$$

The least action principle is established for a free charged particle in a static field to derive the Lorentz force law and the other part of the formalism is the one in which the least action principle is stated for which the actual path is the longest path, namely the Geodesic equation [2].

IV. COVARIANCE OF ELECTRODYNAMICS

To make a clear covariant description of the relativistic Lagrangian, Lorentz force in (14) is written in 4-vector form by introducing electromagnetic field tensor. The form of the Lagrangian for a charged particle in an electromagnetic field suggest that the covariant form of the action integral is [2]

$$\delta\mathrm{S}=-\delta\int\left[\mathrm{mc}(dx_{\alpha}dx^{\alpha})^{\frac{1}{2}}+\frac{e}{c}A_{\alpha}dx^{\alpha}\right]=0 \tag{15}$$

$$\delta\mathrm{S}=-\int\left[\mathrm{mc}\delta(dx_{\alpha}dx^{\alpha})^{\frac{1}{2}}+\frac{e}{c}(\delta A_{\alpha})dx^{\alpha}+\frac{e}{c}A_{\alpha}d(\delta x^{\alpha})\right]=0$$

Applying chain-rule to the first term of the integrand, taking $(dx_{\alpha}dx^{\alpha})^{\frac{1}{2}}=ds=cd\tau$, and $U^{\alpha}=\frac{dx_{\alpha}}{ds}$. Then,

$$\delta(dx_{\alpha}dx^{\alpha})^{\frac{1}{2}}=\frac{dx_{\alpha}\delta(dx^{\alpha})}{(dx_{\alpha}dx^{\alpha})^{\frac{1}{2}}}=\frac{dx_{\alpha}\delta(dx^{\alpha})}{ds}=\frac{dx_{\alpha}\delta(dx^{\alpha})}{cd\tau}=U^{\alpha}d(\delta x^{\alpha})$$

The action integral gives

$$\delta\mathrm{S}=-\int\left[\mathrm{mc}U^{\alpha}d(\delta x^{\alpha})+\frac{e}{c}(\delta A_{\alpha})dx^{\alpha}+\frac{e}{c}A_{\alpha}d(\delta x^{\alpha})\right]=0 \tag{16}$$

Introducing integration by part with some manipulations, the integrand becomes

$$\int\mathrm{mc}\frac{dU^{\alpha}}{\mathrm{ds}}-\frac{e}{c}\left[\frac{\partial A_{\beta}}{dx^{\alpha}}-\frac{\partial A_{\alpha}}{dx^{\beta}}\right]dx_{\alpha}\mathrm{ds}=0 \tag{17}$$

The term in the square bracket is regarded as the ***electromagnetic field tensor*** denoted by $F^{\alpha\beta}=\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha}$. Then, the integral leads to

$$\mathrm{mc}\frac{dU^{\alpha}}{\mathrm{ds}}=\frac{e}{c}\left(\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha}\right) \tag{18}$$

This equation is known to be the equation of motion of the charged particles moving in an electromagnetic field described by the electromagnetic field tensor $F^{\alpha\beta}$. It is a second-rank, anntisymmetric field-strength with six independent components of **E** and **B** in matrix form.

$$F^{\alpha\beta}=\begin{pmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{pmatrix} \tag{19}$$

Lowering the indices $\alpha\beta$ gives $F_{\alpha\beta}$ and its elements are obtained by putting $\boldsymbol{E}\rightarrow-\boldsymbol{E}$ in $F^{\alpha\beta}$ according to the signature metric $(+,-,-,-)$ using in this study.

$$F_{\alpha\beta}=g_{\alpha\gamma}F^{\gamma\delta}g_{\delta\beta}=(-\boldsymbol{E},\boldsymbol{B})$$

$$F_{\alpha\beta}=\begin{pmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{pmatrix} \tag{20}$$

To complete the demonstration of electrodynamics equations in covariant form, Maxwell's equations is a must to derive and be written explicitly in covariance form. To start with, let's consider the situation for which fields are not static. The action integral for such field is scalar and is given as

$$\mathrm{S_F}=\frac{1}{c}\int dx^4\mathcal{L}\left(A^{\alpha},\partial^{\beta}A^{\alpha}\right) \tag{21}$$

In the case of the electromagnetic field theory, the Lorentz action integral in (21) is preserved only if the Lagrangian density $\mathcal{L}$ is scalar [2]. Hence, the only Lorentz invariant for the free-field Lagrangian is of the quadratic form of some multiple of $F_{\alpha\beta}F^{\alpha\beta}$. The matter-field interaction part is described as a multiple current density 4-vector. The electromagnetic field action integral is now the summation of the action integral for free particles in static background fields ($\mathrm{S_f}$), action integral for the free field when particles are fixed or known ($\mathrm{S_F}$), and the action integral for the matter-field interaction ($\mathrm{S_{int}}$).

$$S=S_f+S_F+S_{int} \tag{22}$$

$$S=-\sum mc\int ds-\frac{1}{16\pi c}\int F^{\alpha\beta}F_{\alpha\beta}\,dx^4-\frac{1}{c^2}\int A_{\alpha}J^{\alpha}\,dx^4$$

The first term, which is the free field action integral, has been used to derive the Lorentz force. Considering the last two terms; matter action integral and matter-field action integral with $dx^4=cdtdxdydz$, the electromagnetic Lagrangian density is given as [2, 4, 7]

$$\mathcal{L}=\frac{1}{16\pi}F^{\alpha\beta}F_{\alpha\beta}-\frac{1}{c}A_{\alpha}J^{\alpha} \tag{23}$$

$$\mathcal{L}=-\frac{1}{16\pi}\left(\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha}\right)\left(\partial_{\alpha}A_{\beta}-\partial_{\beta}A_{\alpha}\right)-\frac{1}{c}A_{\alpha}J^{\alpha}$$

$$\mathcal{L} = -\frac{1}{16\pi}\left(\partial^\alpha A^\beta \partial_\alpha A_\beta - \partial^\alpha A^\beta \partial_\beta A_\alpha - \partial^\beta A^\alpha \partial_\alpha A_\beta + \partial^\beta A^\alpha \partial_\beta A_\alpha\right) - \frac{1}{c} A_\alpha J^\alpha$$

The two middle terms are the same as the two outer terms, so the electromagnetic Lagrangian density becomes

$$\mathcal{L} = -\frac{1}{8\pi}\left(\partial^\alpha A^\beta \partial_\alpha A_\beta - \partial^\alpha A^\beta \partial_\beta A_\alpha\right) - \frac{1}{c} A_\alpha J^\alpha \qquad (24)$$

Substituting (24) into the following Euler Lagrange equation of motion

$$\partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)}\right) = \frac{\partial \mathcal{L}}{\partial A_\beta} \qquad (25)$$

The equation of motion for electromagnetic field becomes

$$-\frac{1}{4\pi}\partial_\alpha(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -J^\beta \qquad (26)$$

The quantity in the bracket is the electromagnetic field tensor $F^{\alpha\beta}$ so that the equation becomes

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \qquad (27)$$

where $J^\beta = (J^0, J^i) = (c\rho, \boldsymbol{J})$ for $(i = 1,2,3)$. The above recipes are enough to generate the Maxwell's equation and to verify their consistency.

A. Case 1: Inhomogeneous Maxwell's equations: Consider the indices arrangement of $(\alpha = 1,2,3)$ for $(\beta = 0)$, so

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{4\pi}{c} J^0$$
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi\rho$$
$$\nabla . \boldsymbol{E} = 4\pi\rho \qquad (28)$$

Furthermore, indices are choosing accordingly as $(\alpha = 0,2,3)$ for $(\beta = 1)$, $(\alpha = 0,1,3)$ for $(\beta = 2)$, and $(\alpha = 0,1,2)$ for $(\beta = 3)$.

Then, for $(\alpha = 0,2,3)$ and $(\beta = 1)$

$$\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = 4\pi J^1$$
$$\frac{-\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = 4\pi J^x \qquad (29)a$$

Also, for $(\alpha = 0,1,3)$ and $(\beta = 2)$

$$\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} = 4\pi J^2$$
$$\frac{-\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} = 4\pi J^y \qquad (29)b$$

Also, for $(\alpha = 0,1,2)$ and $(\beta = 3)$

$$\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} = 4\pi J^3$$
$$\frac{-\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 4\pi J^z \qquad (29)c$$

Addition of the (29)a, (29)b, and (29)c gives the Ampere –Maxwell's equation as

$$\nabla x B - \frac{\partial E}{\partial t} = 4\pi J \qquad (30)$$

B. Caese 2: homogeneous Maxwell's equations: the source $\boldsymbol{J} = 0$ in (27) gives

$$\partial_\alpha F^{\alpha\beta} = 0 \qquad (31)$$

In electromagnetic field theory, this will be best described by defining dual tensor $\mathfrak{F}^{\alpha\beta}$ with the help of a pseudotensor [3]. This is achieved by introducing the total anntisymmetric four-rank tensor $\in^{\alpha\beta\gamma\delta}$, also known as Levi-Civita symbol in four dimensions. For any even permutation $\in^{\alpha\beta\gamma\delta} = +1$ $(\alpha = 0,\ \beta = 1, \gamma = 2,\ \delta = 3)$, for any odd permutation $\in^{\alpha\beta\gamma\delta} = -1$ for any odd permutation and $\in^{\alpha\beta\gamma\delta} = 0$ if any two indices are equal. Contracting $F^{\alpha\beta}$ leads to

$$\mathfrak{F}^{\alpha\beta} = \frac{1}{2} \in^{\alpha\beta\gamma\delta} F_{\alpha\beta} \qquad (32)$$

One can obtain the components of the dual tensor by permutation of indices or simply by putting $\boldsymbol{E} \to \boldsymbol{B}$ and $\boldsymbol{B} \to -\boldsymbol{E}$ in $F^{\alpha\beta}$. Therefore, the dual field tensor is defined by

$$\mathfrak{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & 1.-E_y \\ B_y & -E_z & 0 & -B_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \qquad (33)$$

The covariant form of the homogeneous Maxwell's equation is given by

$$\partial_\alpha \mathfrak{F}^{\alpha\beta} = 0 \qquad (34)$$

For $(\alpha = 1,2,3)$ and $(\beta = 0)$

$$\partial_1 \mathfrak{F}^{10} + \partial_2 \mathfrak{F}^{20} + \partial_3 \mathfrak{F}^{30} = 0$$
$$\partial_1 B_x + \partial_2 B_y + \partial_3 B_z = 0$$
$$\boldsymbol{\nabla} . \boldsymbol{B} = 0 \qquad (35)$$

Similar approach used in deriving Ampere's law will be used to derive the second homogeneous Maxwell equation (Faraday's). Since it is known that varying magnetic field produces the electric field and vise versa, the same indices can be adopted as in faraday's law of electromagnetic. Firstly, let's consider $(\alpha = 0,2,3)$ and $(\beta = 1)$

$$\partial_0 \mathfrak{F}^{01} + \partial_2 \mathfrak{F}^{21} + \partial_3 \mathfrak{F}^{31} = 0$$
$$-\partial_0 B_x - \partial_2 E_z + \partial_3 E_y = 0$$

Also, for $(\alpha = 0,1,3)$ and $(\beta = 2)$

$$\partial_0 \mathfrak{F}^{02} + \partial_1 \mathfrak{F}^{12} + \partial_3 \mathfrak{F}^{32} = 0$$
$$-\partial_0 B_y + \partial_1 E_z + \partial_3 E_x = 0$$

Also, for $(\alpha = 0,1,2)$ and $(\beta = 3)$

$$\partial_0 \mathfrak{F}^{03} + \partial_1 \mathfrak{F}^{13} + \partial_2 \mathfrak{F}^{23} = 0$$
$$-\partial_0 B_z - \partial_1 E_y + \partial_2 E_x = 0$$

Addition of the three equations gives the ampere law of electromagnetic

$$\nabla \text{ x } B + \frac{\partial E}{\partial t} = 0 \tag{36}$$

Mostly, homogeneous Maxwell's equations are best described in terms of $F^{\alpha\beta}$ as the four-dimensional equations

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$
$$\frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} = 0 \tag{37}$$

This is the four-dimensional equation in terms of electromagnetic field tensor called ***Bianchi identity***. It is a constraint that any given fields must satisfy before they can be called fields. The conservation of the source current density can be obtained by taking the 4-divergence of both sides of (27)

$$\partial_\beta \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} \partial_\beta J^\beta$$
$$\frac{\partial F^{\alpha\beta}}{\partial x^\beta \partial x^\alpha} = \frac{4\pi}{c} \partial_\beta J^\beta \tag{38}$$

The contraction on the left-hand side vanishes since the differential operator is symmetric and the $F^{\alpha\beta}$ is anntisymmetric. Hence,

$$\frac{4\pi}{c} \partial_\beta J^\beta = \frac{\partial \boldsymbol{J}^\beta}{\partial x^\beta} = 0 \tag{39}$$

This gives the continuity equation

$$\frac{\partial \boldsymbol{J}^0}{\partial x^0} + \frac{\partial \boldsymbol{J}^i}{\partial x^i} = 0 \qquad (i = 1,2,3)$$
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.\boldsymbol{J} = 0 \tag{40}$$

V. CONCLUSION

Starting from the principle of stationary action, all the basic and iconic equations of classical electrodynamics were derived. This was achieved from the construction of a unique Lorentz invariance and appropriate action integrals for the moving particles in static fields, moving fields, and matter-field interaction. These tools were used to derive the most important of classical electrodynamics equations; Lorentz force equation, Maxwell's equations (homogeneous and inhomogeneous), Bianchi identity equation and continuity equation (charge conservation). The four Maxwell's equations were confirmed to be consistent. Using the generated equations of motion of charged particles including the representation of **E** and **B** in terms of scalar and vector potentials, other equations of electrodynamics are easy to obtain. In conclusion, as far as physics is concern, least action principle occupies the central part.

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