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A Note On Archimedian Chained Γ-Semigroups

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Abstract:- In this article, we introduce chained Γ-semigroups, cancellative Γ-semigroups and obtain some equivalent conditions. Also, we prove that if S is a chained Γ-semigroup, then S is an Archimedian Γ-semigroup with no Γ-idempotents if and only if $s^{\omega}\Gamma S$ satisfies the concentric condition for every $s \in S$. Furthermore, we prove that a cancellative Archimedian chained Γ-semigroup is a Γ-group if $s^{\omega}\Gamma S$ does not satisfy the concentric condition for some $s \in S$. Finally, we prove that if S is a chained Γ-semigroup containing cancellable elements. Then, S is a cancellative Γ-semigroup provided $s^{\omega}\Gamma S$ satisfies the concentric condition for every $s \in S$. The converse is true if S is a Noetherian Γ-semigroup without Γ-idempotents.

Keywords:- *Maximal Γ-Ideal, Prime Γ-Ideal, Γ-Radical, Γ-Idempotent, Chained Γ-Semigroup , Archimedian Γ-Semigroup, Noetherian Chained Γ-Semigroup.*

AMS Subject Classification: 20M12, 20M10, 20N99, 20N20, 51M05

I. INTRODUCTION AND DEFINITIONS

The theory of semigroups was developed by Clifford and Preston [3], [17]. Semigroups are omnipresent. Groups are semigroups with a unit and inverses. Rings are double semigroups: an inner semigroup which is required to be a commutative group and an outer semigroup that does not satisfy any additional conditions. The outer semigroup must distribute over the inner one. If the requirement that the inner semigroup have no requirement for inverses, then we have semirings. Semilattices are semigroups whose multiplication is idempotent. Lattices are double semigroups again and we may or may not need distributivity. The reason for the separate term monoid is that the unit is important.

The phrase idempotent first appears in the work of Kolokoltsov and Maslov [36]. Idempotency has appeared from different sources. The idempotent semigroups was introduced by McLean [9]. Kimura studied regular idempotent semigroups as a general class of idempotent semigroups [22]. Schwarz studied prime and maximal ideals in semigroups [30]. Globally idempotent semigroups has been introduced and studied by Lajos et al [31], [32]. Tamura defined and studied commutative archimedean semigroups [35]. Ciric and Bogdanovic [19] studied (completely) 0-Archimedian semigroups as a generalization of (completely) 0-simple (completely) Archimedian semigroups and nil-extensions of (completely) 0-simple semigroups. Bratislava [26] studied prime ideals and maximal ideals in semigroups. Chain rings was introduced and studied by Paul[2]. Hanumanta studied chain ternary semigroups [10]. Gangadhara studied duo chain Γ-semigroup [1]. Radha et al. [11] studied some structures of idempotent commutative semigroup. Ferreroa et al. [20] studied ideal theory of right chain semigroups that he described in terms of prime and completely prime ideals. Changhphas et al. [34] studied the ideal theory of right chain ordered semigroups in terms of prime ideals, completely prime ideals extending to these semigroups results on right chain semigroups proved in Ferrero et al. [20]. Xu Si-jun and Xu Xin-zhai studied some structures, and discriptions on commutative chained ordered semigroups are established mainly from semigroups by Satyanarayana[18]. Maximal Γ-ideal, prime Γ-ideal, Γ-radical, Γ-idempotent, chained Γ-semigroup, archimedian Γ-semigroup, Noetherian chained Γ-semigroup are very important in the generalized structure theory of semigroups. However, these notions have not been studied deeply. The motivation for this paper is to understand the structure of a Γ-semigroup. Thereafter, we study some new types of Γ-semigroups like maximal Γ-ideal, prime Γ-ideal, Γ-radical, Γ-idempotent, chained Γ-semigroup, archimedian Γ-semigroup, Noetherian chained Γ-semigroup.

The notion of Γ-semigroups was defined in 1981 by Sen [12] and then in 1986 by Sen and Saha [13]. Thereafter, these notions of Γ-structures have been studied in numerous papers [14], [15], [16], [21], [23], [24], [27]. Recently, Basar et al. deeply studied and obtained results on Γ-semigroups, ordered Γ-semigroups, Γ-hypersemigroups and hypersemigroups [5], [6], [7], [8], [25].

A Γ-semigroup is ordered triplets $(S,\Gamma,\cdot)$ consisting of two sets S and Γ and a ternary operation $S\times\Gamma\times S\to S$ with the property that $(axb)yc = ax(byc)$ for all $a,b,c\in S$ and $x,y\in\Gamma$. Let A be a nonempty subset of $(S,\Gamma,\cdot)$. Then, A is called a sub-Γ-semigroup of $(S,\Gamma,\cdot)$ if $a\cdot\gamma\cdot b\in A$ for all $a,b\in A$ and $\gamma\in\Gamma$. Furthermore, a Γ-semigroup S is called commutative if $a\cdot\gamma\cdot b = b\cdot\gamma\cdot a$ for all $a,b\in S$ and $\gamma\in\Gamma$. A group S is called Γ-group if $h\alpha S = S\beta g = S$ for all $(h,g)\in S^2$ and for $\alpha,\beta\in\Gamma$. If we consider, $\Gamma=\{1\}$ in the definition, then every semigroup becomes a Γ-semigroup. The terminologies related to chained commutative ternary semigroups, maximal Γ-ideal, prime Γ-ideal, prime Γ-radical, Γ-semigroup, cancellative Γ-semigroup, Archimedian Γ-semigroup, idempotent Γ-semigroup are defined in [28], [29].

Definition 1.1 *[4] Suppose S is a Γ-semigroup. Then, the concentric condition is defined as $s^\omega\Gamma S$ for every $s\in S$ and ω is an infinite ordinal number.*

II. CHAINED Γ-SEMIGROUPS, NOETHERIAN Γ-SEMIGROUPS AND CANCELLATIVE ARCHIMEDIAN CHAINED Γ-SEMIGROUP

In this main part, we prove some results concerning chained Γ-semigroups, Archimedian Γ-semigroups, cancellative Archimedian chained Γ-semigroup and prime Γ-ideal. We start with proving the following equivalent conditions:

Theorem 2.1 *Suppose S is a chained Γ-semigroup. Then, the following assertions are equivalent:*

i. $P=\bigcap_{a=1}^{\infty} a^nP$ for every $a\notin P$, where P is a prime Γ-ideal;
ii. $I=\{s\in S: s^\omega\Gamma S = s\Gamma S\}$ is a prime Γ-ideal or empty, where ω is infinite ordinal number;
iii. Let I be any Γ-ideal. Then, I is a prime Γ-ideal;
iv. Suppose S has no Γ-idempotent. Then, for any $s\in S$, $s^\omega\Gamma S$ satisfies the concentric condition for all $s\in S$, or $s^\omega\Gamma S$ is a prime Γ-ideal. Moreover, suppose S is a cancellative Γ-semigroup with an identity, then for every non-unit a, $s^\omega\Gamma S$ satisfies the concentric condition, or $s^\omega\Gamma S$ is a prime Γ-ideal.

Proof.

(i). It is clear, for if $a\notin P$, then for every positive integer n, $a^n\notin P$, and therefore, $P\subseteq a^n\Gamma S$. This hows that $P = a\Gamma P$. Hence, $P=\bigcap_{a=1}^{\infty} a^nP$.

(ii). It is known fact that in chained Γ-semigroups, every Γ-ideal can have at most one minimal prime divisor. Therefore, $\sqrt{I}$ is a prime Γ-ideal for every Γ-ideal I.

(iii) Straightforward.

(iv) Suppose $s^\omega\Gamma S$ does not satisfy the concentric condition and $x\gamma y\in\Gamma$, where $x,y\notin s^\omega\Gamma S$. Therefore, $x\notin s^m\Gamma S$ and $y\notin s^n\Gamma S$ for some positive integers m and n. So, we may assume that both $x,y\notin s^n\Gamma S$ for some n. Therefore, $s^n\Gamma S$ and $s^n\Gamma S\in y\Gamma S$ and $s^{4n}=s^{2n}\alpha s^{2n}\in(x\beta y)\Gamma S\subseteq s^n\Gamma S$ for $\alpha,\beta\in\Gamma$. This establishes the existence of Γ-idempotent.

Theorem 2.2 *Suppose S is a chained Γ-semigroup S with $S\neq S^2$. Then, the following conditions are true:*

i. $S = s\cup s\Gamma S = S^2\cup s$, $s\notin S^2$, and $S^2 = s\Gamma S$ is the unique maximal Γ-ideal.
ii. Let $a\notin s^\omega\Gamma S$. Then, $a=s^n$ for $n>1$. If $a\in s^\omega\Gamma S$, then $a=s^r$ for some positive integer r, or $a=s^n\gamma s_n$ for $\gamma\in\Gamma$ and $s_n\in s^\omega\Gamma S$ for every positive integer n. If $S\neq N$, then every element in $s^\omega\Gamma S$ is of the form s^r.
iii. If $S\neq N$, then s is a cancellable element and $s^\omega\Gamma S$ is a prime Γ-ideal, or empty.
iv. $S\backslash s^\omega\Gamma S=\{s,s^2,\cdots\}$ or $S\backslash s^\omega\Gamma S=\{s,s^2,\cdots,s^r\}$.

Proof.

(i). As Γ-ideals are comparable and $S\backslash s$ is a maximal Γ-ideal and so, $S\backslash S^2$ contains only one element. By chained condition, $S^2\subseteq s\cup s\Gamma S$, and thus $S^2 = s\Gamma S$. Hence, $S = S^2\cup s = s\cup s\Gamma S$.

(ii). Let $a\notin s^\omega\Gamma S$. As, $s\notin S^2$, $a\neq S$. Since, $a\in s\Gamma S$, we have $a\in s^{n-1}\Gamma S\backslash s^n\Gamma S$ for some positive integer n. Therefore, $a=s^{n-1}\gamma s$ for $\gamma\in\Gamma$, $s\in s\Gamma S$, and therefore, $s=s^n$. If $a\in s^\omega\Gamma S$, then $a=s^n\gamma s_n$ for every positive integer n and $\gamma\in\Gamma$. If some $s_i\notin s^\omega\Gamma S$, from the above, $s_i=x^r$ for some positive integer r and hence, a is of the required form.

(iii). Let s be a non-cancellable element, then $S=s\cup s\Gamma S\subseteq N$ and therefore, $S=N$. Now, suppose $a,b\notin s^\omega\Gamma S$ with $ab\in s^\omega\Gamma S$. By (ii), $a=s^n$ and $b=s^m$ for some positive integers m and n. Therefore, $s^{n+m}\in s^\omega\Gamma S$ and so, $s^{n+m}=s^{n+m+2}$ for some $s\in S$. As s is cancellable, we have $s=s^2\gamma s\in S^2$, which is a contradiction.

(iv). Let s^r be the least power contained in $s^\omega\Gamma S$. Therefore, $s^n\in s^\omega\Gamma S$ for all $n>r$ and so, $S\backslash s^\omega\Gamma S$ contains $s,s^2,\cdots,s^r$.

Therefore, $S\backslash s^\omega\Gamma S=\{s,s^2,\cdots,s^r\}$ by (ii). If no power of x is in $s^\omega\Gamma S$, then, as above, we have $S\backslash s^\omega\Gamma S=\{s,s^2,\cdots\}$. This completes the proof of the Theorem.

In the following Theorem, we prove the necessary and sufficient condition under which a chained Γ-semigroup becomes an Archimedian Γ-semigroup.

Theorem 2.3 *Suppose S is a chained Γ-semigroup. Then, S is an Archimedian Γ-semigroup with no Γ-idempotents if and only if $s^{\omega}\Gamma S$ satisfies the concentric condition for every $s \in S$.*

Proof. Let s^{ω} do not satisfy the concentric condition for some $s \in S$ by Theorem 2.1, $s^{\omega}\Gamma S$ is a prime Γ-ideal. If S is an Archimedian Γ-semigroup without Γ-idempotents, then, $s^{\omega}\Gamma S = S$. Hence, $s \in s^2\Gamma S$, which shows that S has Γ-idempotents and this is a contradiction.

Conversely, suppose $s^{\omega}\Gamma S$ satisfies the concentric condition for every $s \in S$. This shows that S has no Γ-idempotents. Let P be a prime Γ-ideal different from S. There exists an $x \notin P$. As Γ-ideals are linearly ordered, $P \subseteq x\Gamma S$ and therefore, $P = x\Gamma P$ so that $P \subseteq x^{\omega}\Gamma S$. Hence, S has no proper prime Γ-ideals, which shows that S is Archimedian. This completes the proof of the Theorem.

The following Theorem establishes a relationship between a cancellative Archimedian chained Γ-semigroup and a Γ-group.

Theorem 2.4 *A cancellative Archimedian chained Γ-semigroup is a Γ-group if $s^{\omega}\Gamma S$ does not satisfy for some $s \in S$.*

Proof. Let S contain an identity. Then, S is a Γ-group since S is Archimedian. Now, it is sufficient to prove that S has an identity. If S has a Γ-idempotent, this idempotent is the identity by the cancellative condition. Suppose S has no Γ-idempotents. Therefore, S has no idempotents. Therefore, by Theorem 2.1, $s^{\omega}\Gamma S$ is a non-empty prime Γ-ideal and thus, $s^{\omega}\Gamma S = S$ by Archimedian property. Hence, S contains Γ-idempotents as in the proof of Theorem 2.3, which is a contradiction. This completes the proof of the Theorem.

In the following theorem, we describe as to when a chained Γ-semigroup becomes a cancellative Γ-semigroup and conversely.

Theorem 2.5 *Suppose S is a chained Γ-semigroup containing cancellable elements. Then, S is a cancellative Γ-semigroup if $s^{\omega}\Gamma S$ satisfy the concentric condition for every $s \in S$. The converse is true if S is a Noetherian Γ-semigroup without Γ-idempotents.*

Proof. Obviously, N does not satisfy the concentric condition, then $N \subseteq s\Gamma S$ for every cancellable element s. As, N is a prime Γ-ideal, we have $N = s\Gamma S$ and therefore, $N \subseteq s^{\omega}\Gamma S$ satisfy the concentric condition. Let S be a Noetherian cancellative Γ-semigroup without Γ-idempotents. Suppose $s^{\omega}\Gamma S$ does not satisfy the concentric condition for some s. Therefore, by Theorem 2.1, $s^{\omega}\Gamma S$ is a prime Γ-ideal. No power of s belongs to $s^{\omega}\Gamma S$ since, otherwise, we have $s^n \in s^{\omega}\Gamma S \Rightarrow s^n = s^{n+1}$.

for some $b \in S$. Thus, $s = s\gamma b$ for $\gamma \in \Gamma$. This shows that b is Γ-idempotent. Now, $x \in s^{\omega}\Gamma S \Rightarrow x = s^i\gamma s_i, s_i \in S$. As, S is cancellative, we have $s_i = s\gamma s_{i+1}$. Moreover, every $s_i \in s^{\omega}\Gamma S$ as $s^i \notin s^{\omega}\Gamma S$ and $s^{\omega}\Gamma S$ is a prime Γ-ideal. So, $s_i \neq s_{i+1}$ as Γ-idempotents do not exist. By Noetherian condition, the chain $s_1\Gamma S \subseteq s_2\Gamma S \subseteq \cdots$, terminates. Hence, $s_i\Gamma S = s_{i+1}\Gamma S \Rightarrow s_{i+1} = s_{i+1}\gamma t, t \in S$ and therefore, Γ-idempotent exists. This is a contradiction. This completes the proof.

III. CONCLUSION

In this article, we introduced chained Γ-semigroups, cancellative Γ-semigroups and obtained some equivalent conditions. We proved that if S is a chained Γ-semigroup, then S is an Archimedian Γ-semigroup with no Γ-idempotents if and only if $s^{\omega}\Gamma S$ satisfies the concentric condition for every $s \in S$. Furthermore, we prove that a cancellative Archimedian chained Γ-semigroup is a Γ-group if $s^{\omega}\Gamma S$ does not satisfy the concentric condition for some $s \in S$. Finally, we proved that if S is a chained Γ-semigroup containing cancellable elements, then, S is a cancellative Γ-semigroup provided $s^{\omega}\Gamma S$ satisfies the concentric condition for every $s \in S$. The converse part is true if S is a Noetherian Γ-semigroup without Γ-idempotents. The results can further be generalized and extended to other algebraic structures.

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