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On Relationship between a Rotund Norm and Locally Uniformly Rotund Norm in Fre'chet Space

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Abstract:- It is known that a Uniformly Rotund norm implies a Locally Uniformly Rotund norm. The question whether if a Fre'chet space F has a Rotund norm implies it has an equivalently Locally Uniformly Rotund norm is still open and represents one of the most interesting and studied problems. In this paper, we investigate if there exists a direct relationship between a Rotund norm and a Locally Uniformly Rotund norm in Fre'chet space. It is shown that if a norm is Locally Uniformly Rotund in a Fre'chet space then it implies that it is Rotund too in a Fre'chet space. It is also shown that if a Fre'chet space is non-reflexive such that its dual is separable then the norm defined on it is an equivalent norm which is Rotund hence Locally Uniformly Rotund. It is further shown that any separable Fre'chet space that admits an equivalent Locally Uniformly Rotund norm must admit a Rotund norm.

Keywords:- *A Norm; Rotund Norm; Locally Uniformly Rotund Norm; Fre'chet Space; Non-Reflexive Fre'chet Space.*

I. INTRODUCTION

Smith in [1] studied the rotundity properties and gave a summary chart for rotundity which clearly showed that a Uniformly Rotund norm(UR) implies a Locally Uniformly Rotund norm(LUR). The question whether if a Fre'chet space has a Rotund norm(R) implies it has an equivalently LUR is still open. For the notion of Weakly Locally Uniformly Rotund norm(WLUR), only partial results have so far been obtained. Godefroy in [2] showed that a Banach space with a WLUR and Fre'chet differentiable norm must be LUR renormable.

Giovanni and Alfonso in [3] considered some rotundity properties which are extensions of the Uniform rotundity and showed that these properties extends from the Banach space E (or from the conjugate Banach space E^*) to the $L^P(\mu,E)$ or to $(L^P(\mu,E))^*, 1<p<\infty$. They investigated whether the Weak Uniform Rotundity(WUR) of a Banach space E implies that E^* has the Radon-Nikodym property. Moreover three other geometric properties, namely Weak Local Uniform Rotundity(WLUR), Weak* Uniform Rotundity(W*UR) (in a conjugate space) and Weak* Local Uniform Rotundity(W* LUR) were considered and it was shown that they extend from E(or E^*) to $L^P(\mu,E)$ or to $(L^P(\mu,E))^*$. A simple chart was used to show the connections among the above rotundity properties that WUR⇒ W*UR⇒ W*LUR and also WUR⇒WLUR⇒ W* LUR which obviously reduced to WUR⇒ WLUR.

(S,∑, μ) was considered and denoted by $(L^P(\mu,E))^*, 1<p<\infty$, the Lebesgue- Bochner function space of $\mu - equivalence$ classes of strongly measurable functions $f: s \longrightarrow E$ with $\int_S^\infty \| f(s) \|^p \, d\mu, 1<p<\infty$, endowed with the norm $\| f \| = \left(\int_S^\infty \| f(s) \|^p \, d\mu\right)^{1/p}$. It was further denoted by $\|.\|_*$, the norm of the conjugate space $(L^P(\mu,E))^*$ and used an integral representation theorem for the elements of $(L^P(\mu,E))^*$. The most general of the theorem from [4], [5] and [6] was picked and used in the study of the structure of $L^P(\mu,E)$ for the first time.

The technique used by Giovanni and Alfonso in [3] allows us to extend the rotundity conditions concerning Locally Uniformly Rotund norm in Every Direction due to the result of Smith and Turret in [7] to the case of an arbitrary (not necessarily finite) measure space (S, ∑, μ). Moreover, they can be used to prove that the conjugate space $(L^P(\mu,E))^*$ is strictly rotund or Uniformly rotund in Every Direction whenever E^* is.

In our case, the aim is to investigate if there exists a direct relationship between a Rotund norm(R) and a Locally Uniformly Rotund norm(LUR) in a Fre'chet space F. Let us consider the following definitions.

II. RESEARCH METHODOLOGY

➢ ***Definition 2.1*** *[8, Definition 1.0]*

Let F denote either the field of $\mathbb{R}$ *or* $\mathbb{C}$. A norm on a real or complex vector space X is areal-valued function on X whose value at an $x \in X$ is denoted by $\| x \|$ (read as norm of x) and which has the properties:

i) $\| x \| \geq 0 \; \forall \, x \in X$
ii) $\| x \| = 0$ if and only if $x = 0$
iii) $\| \alpha x \| = |\alpha| \, \| x \| \; \forall \, \alpha \in$ F
iv) $\| x + y \| \leq \| x \| + \| y \|$(Triangle inequality)

➢ ***Definition 2.2*** *[9, Definition 1.3]*

A norm on a Fre'chet space F is said to be Rotund(R) or strictly convex if for all $x, y \in S_F$,
$\| x + y \| = 2$ implies $x = y$.

➢ ***Definition 2.3*** *[10, Definition 1.5]*

A Fre'chet space F is a locally convex topological vector space. Trivial example of a Fre'chet space is a Banach space. Every Banach space in particular (l^P $\forall 1 \leq p < \infty$) is a Fre'chet space.

➢ ***Definition 2.4*** *[11, Definition 1.2]*

Let $(X, \|.\|)$ be a Banach space. $(X, \|.\|)$ is Uniformly Rotund(UR) if given $\varepsilon > 0$, there exists $\delta > 0$ such that $\| \frac{x+y}{2} \| \leq 1 - \delta$, whenever $\| x - y \| \geq \varepsilon$ and $x, y \in S_X$.

➢ ***Definition 2.5*** *[12, Definition 7]*

Let $(X, \|.\|)$ be a Banach space. The norm, $\|.\|$ of X is Locally Uniformly Rotund(LUR) if for $x, \in S_X$ and any sequence $\{x_n\}$ $\forall$ n $\in$ B_X such that $\lim_{n\to\infty} \| \frac{x_n+x}{2} \| = 1$, we have, $\lim_{n\to\infty} \| x_n - x \| = 0$.

III. RESULTS AND DISCUSSION

In this section, we give the results. We begin with the following theorem.

➢ ***Theorem 3.1***

Let $(F, \|.\|_E)$ be a Fre'chet space. Then the norm $\|.\|_E$ is Locally Uniformly Rotund(LUR).

Proof.

From the hypothesis, $\|.\|_E$ is LUR. We need to show that $\|.\|_E$ is Rotund(R). Let f and $\{f_n\}$ be given such that $\| f \|_E = 1$, $\| f_n \|_E \to 1$ and $\| f + f_n \|_E \to 2$. It follows that

$2(\| f \|_E^2 + \| f_n \|_E^2) - \| f + f_n \|_E^2 \to 0$ and hence

$2(\| f \|_1^2 + \| f_n \|_1^2) - \| f + f_n \|_1^2 \to 0$.................(3.1.1)

and

$2(\| I f \|_2^2 + \| I f_n \|_2^2) - \| I f + I f_n \|_2^2 \to 0$...................(3.1.2).

From equation (3.1.1) , it follows that $\| f_n \|_1 \to \| f \|_1$ and from equation (3.1.2) that

$f_n^i \to f^i$ $\forall$ i . Together these two conditions are sufficient to imply that $f_n \to f$ hence $\|.\|_E$ is Rotund.

➢ ***Theorem 3.2***

Let $(E, \|.\|)$ be a non- reflexive Fre'chet space such that E^* is separable. Let $\|.\|$ be norm on E and $\{f_n\}_{n\in\mathbb{N}}$ be dense of S_{E^*}. The norm $|.|$ defined by $|e|^2 = \| e \|^2 + \sum_{\geq i} 2^{-i} f_i^2(e)$, $\forall e \in E$ is an equivalent norm on E such that the norm $|.|^{**}$ on E^{**} is Rotund(R).

Proof

From the hypothesis, the equivalent norm $|.|$ is Rotund on E such that $|.|^{**}$ is also Rotund in E^{**}. We need to show that $|.|^{**}$ is LUR in E^{**}. Let a^{**}, $b^{**} \in S_{E^{**}}$ such that the norm

$$|a^{**}|^{**} = |b^{**}|^{**} = |\frac{a^{**} + b^{**}}{2}|^{**} = 1.$$

Since $\varphi(B_E)$ is $w^* - dense$ in $B_{E^{**}}$ there exists two nets $\{a_\alpha\}_{\alpha\in\Gamma} \subseteq B_E$ and $\{b_\alpha\}_{\alpha\in\Gamma} \subseteq B_E$ such that $a_\alpha \to a^{**}$, $b_\alpha \to b^{**}$ in $w^* - topology$, then $\widehat{a_\alpha} + \widehat{b_\alpha} \to a^{**} + b^{**}$ in $w^* - topology$. The norm $|.|^{**}$ is $w^* - lower$ semicontinuous then for arbitrary $\varepsilon > 0, \exists\, \alpha_0 \in \Gamma: \forall\, \alpha > \alpha_0$ we have

$$2 - \varepsilon < \left|\widehat{a_\alpha} + \widehat{b_\alpha}\right|^{**} = |a_\alpha + b_\alpha| \leq 2,$$

Therefore $|a_\alpha + b_\alpha| \to 2$ that implies $2|\widehat{a_\alpha}|^2 + 2|\widehat{b_\alpha}|^2 - |a_\alpha + b_\alpha|^2 \to 0$ or

$2 \| a_\alpha \|^2 + 2\sum_{\geq i} 2^{-i} f_i^2(a_\alpha) + 2 \| b_\alpha \|^2 + 2\sum_{\geq i} 2^{-i} f_i^2(b_\alpha) - \| a_\alpha + b_\alpha \|^2 - \sum_{\geq i} 2^{-i} f_i^2(a_\alpha + b_\alpha) \to 0$. For every $\alpha \in \Gamma$ and every $n \in \mathbb{N}$ we have

$$2 \| a_\alpha \|^2 + 2 \| b_\alpha \|^2 - \| a_\alpha + b_\alpha \|^2 \geq 0,$$

$$2f_n^2(a_\alpha) + 2f_n^2(b_\alpha) - f_n^2(a_\alpha + b_\alpha) = f_n^2(a_\alpha - b_\alpha) \geq 0.$$

Therefore $f_n(a_\alpha - b_\alpha) \to 0$ $\forall$ $n \in \mathbb{N}$. Since $\{f_n\}_{n\in\mathbb{N}}$ is dense in S_{E^*} we have $f(a_\alpha - b_\alpha) \to 0$ for every $f \in \{e_i\}_{i=1}^{\infty}$ and consequently $a^{**} = b^{**}$. Therefore $|.|^{**}$ is Rotund in E^{**}.

➢ ***Theorem 3.3***

- If $(E, \|.\|)$ is any separable Fre'chet space, then E admits an equivalent Locally Uniformly Rotund(LUR) norm.
- If E^* is separable then E admits an equivalent norm whose dual norm is Locally Uniformly Rotund(LUR).

Proof

From the hypothesis in part (i) and (ii), it is clear that E and E^*admit an equivalent Locally Uniformly Rotund(LUR) norms in Fre'chet space given that they are separable. We want to show that they can admit a Rotund norm too.

Suppose that $\{e_i\}_{i=1}^{\infty}$ be dense in S(E) and $\{f_i\}_{i=1}^{\infty}$ be dense in $S(E^*)$. For $i \in \mathbb{N}$, put

$F_i = span\{f_{1,2,........} f_i\}$ define a norm $|\ \|.\|\ |$ on E^* by $|\ \| f \|\ |^2 = \| f \|^{*2} + \sum_{i=1}^{\infty} 2^{-i}\, dist\,(f, F_i)^2 + \sum_{i=1}^{\infty} 2^{-i} f^2(e_i)$. $|\ \|.\|\ |$ is a weak* lower semicontinous function on E^* equivalent with $\|.\|^*$. Hence $|\ \|.\|\ |$ is the dual of a norm $|.|$ equivalent with $\|.\|$ and also it is Rotund.

IV. CONCLUSIONS

Rotundity in Fre'chet spaces has been studied over a period of time. Various notions of rotundity have been considered with very interesting results obtained [1-6]. Let $(F, \|.\|)$ be a Fre'chet space and trivial example of a Fre'chet space is a Banach space. Every Banach space in particular (l^P $\forall 1 \leq p < \infty$) is a Fre'chet space. We have shown that if a norm is Locally Uniformly Rotund in a Fre'chet space then it implies that it is Rotund too in a Fre'chet space. It is also shown that if a Fre'chet space is non-reflexive such that its dual is separable then the norm defined on it is an equivalent norm which is Rotund hence Locally Uniformly Rotund. It is further shown that any separable Fre'chet space that admits an equivalent Locally Uniformly Rotund norm must admit a Rotund norm.

➢ ***Conflict of Interest***
The author declares no conflict of interest.

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