Generalised Inner Derivations in Semi Prime Rings

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Abstract:- Let A be any ring and $f(xy) = f(x)y+xh_a(y)$, where f be any generalised inner derivation(G.I.D) a be the fixed element of A.

In this paper, it is shown that (i) h_a must necessarily be a derivation for semi prime ring A. (ii) ∃ no generalized inner derivations $f : A \to A$ such that

$$f(x \circ y) = x \circ y$$

or

$$f(x \circ y) + x \circ y = 0 \ \forall x,y \in A,$$

We have proved Havala [2] def. p.1147, Herstein [3] Lemma 3.1 p. 1106 as corollaries, along with other results.

I. INTRODUCTION

We have defined the G.I.D. of a ring. "Let A be any ring. An additive mapping $f : A \to A$ is said to be G.I.D if $f(xy) = f(x)y + xh_a(y)$

where $h_a : A \to A \ y \to [a,y] \ \forall x,y \in A$, fixed element $a \in A$," Let $G_ID(A)$ be the set of all G.I.D of A into itself. We prove that $f(xyz) = f(x)yz + xh_a(yz)$, $\forall x,y,z \in A$, $f \in G_ID(A)$.

In Theorem 2.2, we have shown that h_a must necessarily be a derivation on A where A be any semi prime ring. In Corollary 2.3, replacing h_a by d, we get Havala [2] def. p-1147 of Generalized derivation. In Corollary 2.5 replacing f by d, we get Herstein [3] Lemma 3.1, P-1093.

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A. In Theorem 3.2, we have proved that ∃ no G.I.D $f : A \to A$ such that

$$f(x \circ y) = x \circ y$$

or

$$f(x \circ y) + x \circ y = 0 \ \forall x,y \in A$$

and $x \circ y = xy + yx$. , For any 2 torsion free semi prime ring A with identity .

Finally in Theorem 4.1 we have proved that $(1+f(1))[x,y]+[a,[x,y]] = 0$ for any non-zero Ideal K of A with unity.

➢ *Generalized Inner Derivation*

In this section, we study the G.I.D in a ring.

Definition 1.1. (Generalized Inner Derivation): An additive mapping $f : A \to A$ is said to be G.I.D if $f(xy) = f(x)y + x[a,y]$, for fixed element $a \in A$ and $\forall x,y \in A$

We are taking the definition as
$f(xy) = f(x)y + xh_a(y)$

where
$h_a : A \to A \ y \to [a,y]$ is the inner derivation.
Let $G_ID(A)$ be the set of all G.I.D of A into itself.

Lemma 1.2 *If $f \in G_ID(A)$. Then $f(xyz) = f(x)yz + xh_a(yz), \forall x,y,z \in A$.*

Proof. Now

$$\begin{aligned} f(xyz) &= f(xy)z + xy[a,z], \\ &= (f(x)y + x[a,y])z + xy[a,z] \\ &= f(x)yz + x[a,y]z + xy[a,z] \\ &= f(x)yz + x(ay - ya)z + xy(az - za) \\ &= f(x)yz + xayz - xyaz + xyaz - xyza \\ &= f(x)yz + x(ayz - yza) \\ &= f(x)yz + x[a,yz] \\ f(xyz) &= f(x)yz + xh_a(yz) \end{aligned}$$

Hence proved.

B. In this section, we take A be any semi prime ring.

Definition 2.1. (Semi prime ring) : Let A be any ring. Then A is said to be semi-prime ring if $xax = 0 \ \forall a \in A \Rightarrow x = 0$

Now A = semi prime ring has the following property:
"If $Aa = 0$, $a \in A$ then $a = 0$"

This result is used in Theorem 2.2.

Theorem 2.2 *Let A be any semi-prime ring and f be a generalized inner derivation of A. Then h_a must neccessarily be a derivation.*

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Proof. Now

$f(xyz) = f(x)yz + xh_a(yz).$ (1)

Also

$$f(xyz) = f(xy)z + xy\, h_a(z) = (f(x)y + xh_a(y))z + xy\, h_a(z) = f(x)yz + xh_a(y)z + xyh_a(z) \quad (2)$$

From (1) and (2), we get

$x(h_a(yz) - h_a(y)z - yh_a(z)) = 0$ (3)

Now

$$f(x(y+z)) = f(x)(y+z) + xh_a(y+z) = f(x)y + f(x)z + xh_a(y+z). \quad (4)$$

Also

$$f(x(y+z)) = f(xy+yz) = f(xy) + f(xz) = f(x)y + xh_a(y) + f(x)z + xh_a(z) \quad (5)$$

From (4) and (5), we have

$x(h_a(y+z) - h_a(y) - h_a(z)) = 0$ (6)

Since A is semi prime ring and if $Aa = 0$ Then $a = 0$, $a \in A$.
⇒ From (3) and (6), we have

$$h_a(yz) - h_a(y)z - yh_a(z) = 0$$
$$h_a(y+z) - h_a(y) - h_a(z) = 0$$

$\Rightarrow h(yz) = h_a(y)z + yh_a(z)$ $h_a(y+z) = h_a(y) + h_a(z)$

Hence h_a is a derivation. Hence proved.

Corollary 2.3 $f(xy) = f(x)y + xh_a(y)$ where h_a is a derivation replacing h_a by d, we get Havala [2] def. P.1147 of Generalized derivation

Corollary 2.4 Havala [2] result is also proved. "Let A be any semi prime ring. Then $f(xyz) = f(x)yx + xyd(x) + xd(y)x$ ∀ $x,y \in A$
where f is generalized derivation of A. "

Corollary 2.5 Replacing f by d in Corollary 2.4, we get Lemma 3.1
P.1106 of Herstein [3]
$d(xyz) = d(x)yx + xd(y)x + xy\, d(x)$

C. In this section, we take A be any 2 torsion free semi-prime ring.

Definition 3.1. Let A be any ring which is 2 torsion free and also semi prime. We define
$x \circ y = xy + yx$ ∀ $x,y \in A$

Theorem 3.2 Let A be any ring which is 2 torsion free and also semi prime with Identity 1. Then ∃ no generalized inner derivation $f : A \to A$ such that

$$f(x \circ y) = x \circ y \ \forall\, x,y \in A$$

or $f(x \circ y) + x \circ y = 0.$

Proof. If possible, let ∃ a generalized derivation $f : A \to A$ s.t $f(x \circ y) = x \circ y$
or $f(x \circ y) + x \circ y = 0$ ∀ $x,y \in A$.

Now

$$f(x \circ y) = x \circ y$$

or

$$f(x \circ y) + x \circ y = 0$$

Putting $y = 1$

$\Rightarrow f(x \circ 1) = x \circ 1$ or $f(x \circ 1) + x \circ 1 = 0$
$\Rightarrow f(x + x) = x + x$ or $f(x + x) + x + x = 0$
$\Rightarrow f(2x) = 2x$ or $f(2x) + 2x = 0$
$\Rightarrow f(2x) = 2x$ or $f(2x) = -2x$
$\Rightarrow f(2x) = \pm 2x$ ∀ $x \in A$&A is 2-Torsion free.

Now $xy + yx = x \circ y = \pm f(x \circ y)$
$= \pm f(xy + yx)$
$= (-(xy + yx))$ ($\because f(x \circ y) = -(x \circ y)$) $xy + yx$
$= \mp(xy + yx)$
$\Rightarrow 2(xy + yx) = 0$
$\Rightarrow x \circ y = 0$

Since A is 2-Torsion free
$\Rightarrow x^2 = 0$

Now
$x \circ y = 0$
$\Rightarrow x \circ (x + 1) = 0$ (Taking $y = x + 1$)
$\Rightarrow 2x = 0$
$\Rightarrow x = 0$ ∀ $x \in A$

which is a contradiction (∵ Identity is 1).Hence our supposition is wrong.
So, ∃ no generalized inner derivation f satisfying

$$f(x \circ y) = x \circ y$$

or

$$f(x \circ y) + x \circ y = 0 \quad \forall\, x,y \in A$$

Hence proved.

D. In this section, we take A be any ring with unity and we consider non-zero Ideal of A

Lemma 4.1 Let K 6= {0} be an Ideal of A with unity and f be a

G.I.D of *A*. Then $\forall\ x,y \in K$ satisfying $xy + f(xy) = yx + f(yx)$
$(1 + f(1))[x,y] + [a,[x,y]] = 0$

Proof.

$xy + f(xy) = yx + f(yx)$

$\Rightarrow xy + f(x)y + xh_a(y) = yx + f(y)x + yh_a(x)$

$\Rightarrow xy + (f(1)x + 1h_a(x))y + xh_a(y) = yx + (f(1)y + 1h_a(y))x + yh_a(x)$

$\Rightarrow xy + f(1)xy + axy - xya = yx + f(1)yx + ayx - yxa$

$\Rightarrow (xy - yx) + f(1)xy - f(1)yx + axy - xya - ayx + yxa = 0$

$\Rightarrow [x,y] + f(1)[x,y] + a(xy - yx) - (xy - yx)a = 0$

$\Rightarrow [x,y] + f(1)[x,y] + a[x,y] - [x,y]a = 0$

$\Rightarrow [x,y] + f(1)[x,y] + [a,[x,y]] = 0$

$\Rightarrow (1 + f(1))[x,y] + [a,[x,y]] = 0$ Hence proved.

II. CONCLUSION

We showed that for any Ring A which is semi prime , h_a is a derivation . We also proved Havala [2] def. p.1147, Herstein [3] Lemma 3.1 p. 1106 as corollaries of our results.

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