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Uniform Higher Order 6, 7-Point Block Methods for Direct Integration of First Order Ordinary Differential Equations

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Abstract:- This research article focuses on proposing uniform higher order 6,7-point BBDF for the numerical integration of first order ODEs. These methods are formulated via interpolation and collocation techniques using power series as the basis function. Usual properties such zero and absolute stabilities, convergence, order and error constant of the methods have been investigated. The methods were applied to some selected test problems and compared with some existing methods such as BBDF(4), BBDF(5), DIBBDF, SDIBBDF, DI2BBDF, NDISBBDF, 3PVSBBDF, Ode15s and Ode23s to prove the accuracy of the methods. Test performance showed that the new methods are viable.

Keywords:- *Backward differentiation formula, uniform order, block methods.*

I. INTRODUCTION

In years past, different approaches have been used to find numerical approximation to difficult problems in differential equations arising from various fields of study such as chemical engineering, biological sciences, petroleum engineering, physics, e.t.c. Among such approaches is the use of predictor-corrector method. But this approach did not in any way ease better solutions as more functions evaluations prevail in the iteration processes. Thus, increased computational burden. Hence, the need for better and easily implemented methods of solutions with reduced functions evaluations. Block methods were introduced to solve the drawback in predictor-corrector techniques. Block methods preserve the traditional advantage of being self-starting and permitting easy change of step length (Lambert, 1973). Notable among researchers who have developed block methods are (Milne, 1953), (Sagir, 2014) developed a discrete linear multistep method of uniform order for solving first order IVPs, Mohammed and Yahaya (2010), developed fully implicit four point block method of order four for solving first order ordinary differential equations through interpolation and collocation techniques using power series expansion, Odekunle, Adesanya and Sunday (2012), also formulated 4-point block method of order five for solving first order ordinary differential equations through interpolation and collocation approaches using a combination of power series and exponential function, among others. In this research paper, we propose uniform higher order 6,7-point block formula for the solution of first order ordinary differential equation of the form:

$$y' = f(x,y),\ x \in [a,b],\ y(a) = \eta \tag{1}$$

where, f is continuous and differentiable. However, f is assumed to satisfy Lipchitz condition and the existence and uniqueness theorem within the interval of $[0,1]$. The system (1) can be regarded as stiff if its exact solution contains very fast and as well as very slow components (Dahlquist, 1974).

In this research paper, we intend to formulate super block methods with higher order that give better approximations to first order ordinary differential equations than some selected existing numerical methods. Formulation of the methods is briefly explained in section 2. In section 3, the stability properties of the methods are discussed. The performances of the method on some stiff problems is presented in comparison to some existing methods in section 4. Section 5 presents discussion of numerical results and a conclusion is made in the last section.

II. FORMULATION OF THE METHOD

In this section we present the derivation of a uniform higher 6,7-point block methods which is self-starting for solving (1). For better numerical approximation, we shall derive the methods using power series polynomial as the approximate solution given as:

$$y(x) = \sum_{j=0}^{k} a_j x^j \tag{2}$$

From the first derivative of (2), we get:

$$y'(x) = \sum_{j=0}^{k} j a_j x^{j-1} = f_{n+i}, i = (0,1,2,3,4,5,6) \tag{3}$$

where $a_{j's}$ are parameters to be determined. Thus, we interpolate (2) and collocate (3) at $x_{n+j},\ j=0$ and $x_{n+i},\ i=0,1,2,3,4,5,6$ respectively to give the following system of equation using Maple soft environment:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_n+2h & 3(x_n+h)^2 & 4(x_n+h)^3 & 5(x_n+h)^4 & 6(x_n+h)^5 & 7(x_n+h)^6 \\ 0 & 1 & 2x_n+4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 & 5(x_n+2h)^4 & 6(x_n+2h)^5 & 7(x_n+2h)^6 \\ 0 & 1 & 2x_n+6h & 3(x_n+3h)^2 & 4(x_n+3h)^3 & 5(x_n+3h)^4 & 6(x_n+3h)^5 & 7(x_n+3h)^6 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & 4(x_n+4h)^3 & 5(x_n+4h)^4 & 6(x_n+4h)^5 & 7(x_n+4h)^6 \\ 0 & 1 & 2x_n+10h & 3(x_n+5h)^2 & 4(x_n+5h)^3 & 5(x_n+5h)^4 & 6(x_n+5h)^5 & 7(x_n+5h)^6 \\ 0 & 1 & 2x_n+12h & 3(x_n+6h)^2 & 4(x_n+6h)^3 & 5(x_n+6h)^4 & 6(x_n+6h)^5 & 7(x_n+6h)^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} \quad (4)$$

Solving for $a_{j's}$ in (4) and when substituted into (2) gives the first continuous block implicit scheme of the form:

$$y(x)=\sum_{q}\alpha_q y_{n+q}+h\sum_{i=0}^{l}\beta_i f_{n+i} \quad (5)$$

where, α_j and β_j are constants; we assume that $\alpha_k \neq 0$ and that not both α_0 and β_0 are zero and that in particular, $q=0$ and $l=6$ in (5), to get the coefficients in (5) and upon substitution into (5), gives the first scheme and the rest of the schemes are gotten by interpolating and collocating at $x_{n+j},\ j=1,2,3,4,5$ and $x_{n+i},\ i=0,1,2,3,4,5,6$, also solving system of equations in a Maple soft environment to give expressions in the form (5). Thus, we have the following discrete 6-point BBDF:

$$\left.\begin{aligned}
y_{n+1} &= y_n+\frac{19087}{60480}hf_n+\frac{2713}{2520}hf_{n+1}-\frac{15487}{20160}hf_{n+2}+\frac{586}{945}hf_{n+3}-\frac{6737}{20160}hf_{n+4}+\frac{263}{2520}hf_{n+5}-\frac{863}{60480}hf_{n+6}\\
y_{n+2} &= y_{n+1}-\frac{863}{60480}hf_n+\frac{349}{840}hf_{n+1}+\frac{5221}{6720}hf_{n+2}-\frac{254}{945}hf_{n+3}+\frac{811}{6720}hf_{n+4}-\frac{29}{840}hf_{n+5}+\frac{271}{60480}hf_{n+6}\\
y_{n+3} &= y_{n+2}+\frac{271}{60480}hf_n-\frac{23}{504}hf_{n+1}+\frac{10273}{20160}hf_{n+2}+\frac{586}{945}hf_{n+3}-\frac{2257}{20160}hf_{n+4}+\frac{67}{2520}hf_{n+5}-\frac{191}{60480}hf_{n+6}\\
y_{n+4} &= y_{n+3}-\frac{191}{60480}hf_n+\frac{67}{2520}hf_{n+1}-\frac{2257}{20160}hf_{n+2}+\frac{586}{945}hf_{n+3}+\frac{10273}{20160}hf_{n+4}-\frac{23}{504}hf_{n+5}+\frac{271}{60480}hf_{n+6}\\
y_{n+5} &= y_{n+4}+\frac{271}{60480}hf_n-\frac{29}{840}hf_{n+1}+\frac{811}{6720}hf_{n+2}-\frac{254}{945}hf_{n+3}+\frac{5221}{6720}hf_{n+4}+\frac{349}{840}hf_{n+5}-\frac{863}{60480}hf_{n+6}\\
y_{n+6} &= y_{n+5}-\frac{863}{60480}hf_n+\frac{263}{2520}hf_{n+1}-\frac{6737}{20160}hf_{n+2}+\frac{586}{945}hf_{n+3}-\frac{15487}{20160}hf_{n+4}+\frac{2713}{2520}hf_{n+5}+\frac{19087}{60480}hf_{n+6}
\end{aligned}\right\} (6)$$

Similarly, we obtain the uniform order 7-point BBDF by interpolating and collocating at $x_{n+j},\ j=0$ and $x_{n+i},\ i=0,1,2,3,4,5,6,7$ to give system of equations and solving the system of equations gives $a_{j's}$ and upon substitution in (5) gives the first scheme; also interpolating and collocating at $x_{n+j},\ j=1,2,3,4,5$ and $x_{n+i},\ i=0,1,2,3,4,5,6,7$, to give in (5) to respectively give:

$$\left.\begin{aligned}
&y_{n+1}=y_n+\frac{5257}{17280}hf_n+\frac{139849}{120960}hf_{n+1}-\frac{4511}{4480}hf_{n+2}+\frac{123133}{120960}hf_{n+3}-\frac{88547}{120960}hf_{n+4}\\
&+\frac{1537}{4480}hf_{n+5}-\frac{11351}{120960}hf_{n+6}+\frac{275}{24192}hf_{n+7}\\
&y_{n+2}=y_{n+1}-\frac{275}{24192}hf_n+\frac{5311}{13440}hf_{n+1}+\frac{11261}{13440}hf_{n+2}-\frac{44797}{120960}hf_{n+3}+\frac{2987}{13440}hf_{n+4}\\
&-\frac{1283}{13440}hf_{n+5}+\frac{2999}{120960}hf_{n+6}-\frac{13}{4480}hf_{n+7}\\
&y_{n+3}=y_{n+2}+\frac{13}{4480}hf_n-\frac{4183}{120960}hf_{n+1}+\frac{6403}{13440}hf_{n+2}+\frac{9077}{13440}hf_{n+3}-\frac{20227}{120960}hf_{n+4}\\
&+\frac{803}{13440}hf_{n+5}-\frac{191}{13440}hf_{n+6}+\frac{191}{120960}hf_{n+7}\\
&y_{n+4}=y_{n+3}-\frac{191}{120960}hf_n+\frac{1879}{120960}hf_{n+1}-\frac{353}{4480}hf_{n+2}+\frac{68323}{120960}hf_{n+3}+\frac{68323}{120960}hf_{n+4}\\
&-\frac{353}{4480}hf_{n+5}+\frac{1879}{120960}hf_{n+6}-\frac{191}{120960}hf_{n+7}\\
&y_{n+5}=y_{n+4}+\frac{191}{120960}hf_n-\frac{191}{13440}hf_{n+1}+\frac{803}{13440}hf_{n+2}-\frac{20227}{120960}hf_{n+3}+\frac{9077}{13440}hf_{n+4}\\
&+\frac{6403}{13440}hf_{n+5}-\frac{4183}{120960}hf_{n+6}+\frac{13}{4480}hf_{n+7}\\
&y_{n+6}=y_{n+5}-\frac{13}{4480}hf_n+\frac{2999}{120960}hf_{n+1}-\frac{1283}{13440}hf_{n+2}+\frac{2987}{13440}hf_{n+3}-\frac{44797}{120960}hf_{n+4}\\
&+\frac{11261}{13440}hf_{n+5}+\frac{5311}{120960}hf_{n+6}-\frac{275}{24192}hf_{n+7}\\
&y_{n+7}=y_{n+6}+\frac{275}{24192}hf_n-\frac{11351}{120960}hf_{n+1}+\frac{1537}{4480}hf_{n+2}-\frac{88547}{120960}hf_{n+3}+\frac{123133}{120960}hf_{n+4}\\
&-\frac{4511}{4480}hf_{n+5}+\frac{139849}{120960}hf_{n+6}+\frac{5257}{17280}hf_{n+7}
\end{aligned}\right\} \quad (7)$$

Hence, (6) and (7) represent the proposed uniform higher order direct 6,7-point block formula (D6PBBDF and D7PBBDF) for the numerical solution of first order ordinary differential equations.

- *Stability analysis of the methods*

Let us begin the stability analysis of the methods by first consider the basic definitions below given by Suleiman, Musa, Ismail, Senu and Ibrahim (2014).

Definition 1: A linear multistep method (LMM) is said to be zero stable if no root of the first characteristic polynomial has modulus greater than one and that any root with modulus one is simple (that is, not repeated).

Definition2: A linear multistep method (LMM) is said to be A-stable if its stability region covers the entire (negative) complex half-plane.

Equations (6) and (7) can be rewritten in matrix form as:

$$\left.\begin{array}{l}
\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&1&1&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&1&1&0\\ 0&0&0&0&1&1 \end{pmatrix}\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\\ y_{n+6} \end{pmatrix}=\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{pmatrix}\begin{pmatrix} y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n} \end{pmatrix}\\
+h\begin{pmatrix} 0&0&0&0&0&\frac{19087}{60480}\\ 0&0&0&0&0&-\frac{863}{60480}\\ 0&0&0&0&0&\frac{271}{60480}\\ 0&0&0&0&0&-\frac{191}{60480}\\ 0&0&0&0&0&\frac{271}{60480}\\ 0&0&0&0&0&-\frac{863}{60480} \end{pmatrix}\begin{pmatrix} f_{n-5}\\ f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_{n} \end{pmatrix}\\
+h\begin{pmatrix} \frac{2713}{2520}&-\frac{15487}{20160}&\frac{586}{945}&-\frac{6737}{20160}&\frac{263}{2520}&-\frac{863}{60480}\\ \frac{349}{840}&\frac{5221}{6720}&-\frac{254}{945}&\frac{811}{6720}&-\frac{29}{840}&\frac{271}{60480}\\ -\frac{23}{504}&\frac{10273}{20160}&\frac{586}{945}&-\frac{2257}{20160}&\frac{67}{2520}&-\frac{191}{60480}\\ \frac{67}{2520}&-\frac{2257}{20160}&\frac{586}{945}&\frac{10273}{20160}&-\frac{23}{504}&\frac{271}{60480}\\ -\frac{29}{840}&\frac{811}{6720}&-\frac{254}{945}&\frac{5221}{6720}&\frac{349}{840}&-\frac{863}{60480}\\ \frac{263}{2520}&-\frac{6737}{20160}&\frac{586}{945}&-\frac{15487}{20160}&\frac{2713}{2520}&\frac{19087}{60480} \end{pmatrix}\begin{pmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\\ f_{n+6} \end{pmatrix}
\end{array}\right\} \qquad (8)$$

$$
\left.
\begin{aligned}
&\begin{pmatrix}
1&0&0&0&0&0&0\\
1&1&0&0&0&0&0\\
0&1&1&0&0&0&0\\
0&0&1&1&0&0&0\\
0&0&0&1&1&0&0\\
0&0&0&0&1&1&0\\
0&0&0&0&0&1&1
\end{pmatrix}
\begin{pmatrix} y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\\ y_{n+6}\\ y_{n+7} \end{pmatrix}
=
\begin{pmatrix}
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1\\
0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix} y_{n-6}\\ y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n} \end{pmatrix}\\
&+h\begin{pmatrix}
0&0&0&0&0&0&\frac{5257}{17280}\\
0&0&0&0&0&0&-\frac{275}{24192}\\
0&0&0&0&0&0&\frac{13}{4480}\\
0&0&0&0&0&0&-\frac{191}{120960}\\
0&0&0&0&0&0&\frac{191}{120960}\\
0&0&0&0&0&0&-\frac{13}{4480}\\
0&0&0&0&0&0&\frac{275}{24192}
\end{pmatrix}
\begin{pmatrix} f_{n-6}\\ f_{n-5}\\ f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_{n} \end{pmatrix}\\
&+h\begin{pmatrix}
\frac{139849}{120960}&-\frac{4511}{4480}&\frac{123133}{120960}&-\frac{88547}{120960}&\frac{1537}{4480}&-\frac{11351}{120960}&\frac{275}{24192}\\
\frac{5311}{13440}&\frac{11261}{13440}&-\frac{44797}{120960}&\frac{2987}{13440}&-\frac{1283}{13440}&\frac{2999}{120960}&-\frac{13}{4480}\\
-\frac{4183}{120960}&\frac{6403}{13440}&\frac{9077}{13440}&-\frac{20227}{120960}&\frac{803}{13440}&-\frac{191}{13440}&\frac{191}{120960}\\
\frac{1879}{120960}&-\frac{353}{4480}&\frac{68323}{120960}&\frac{68323}{120960}&-\frac{353}{4480}&\frac{1879}{120960}&-\frac{191}{120960}\\
-\frac{191}{13440}&\frac{803}{13440}&-\frac{20227}{120960}&\frac{9077}{13440}&\frac{6403}{13440}&-\frac{4183}{120960}&\frac{13}{4480}\\
\frac{2999}{120960}&-\frac{1283}{13440}&\frac{2987}{13440}&-\frac{44797}{120960}&\frac{11261}{13440}&\frac{5311}{13440}&-\frac{275}{24192}\\
-\frac{11351}{120960}&\frac{1537}{4480}&-\frac{88547}{120960}&\frac{123133}{120960}&-\frac{4511}{4480}&\frac{139849}{120960}&\frac{5257}{17280}
\end{pmatrix}
\begin{pmatrix} f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\\ f_{n+6}\\ f_{n+7} \end{pmatrix}
\end{aligned}
\right\}
\quad (9)
$$

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Equation (8) and (9) can be rewritten as:

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \tag{10}$$

where,

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}; A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{19087}{60480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{863}{60480} \\ 0 & 0 & 0 & 0 & 0 & \frac{271}{60480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{191}{60480} \\ 0 & 0 & 0 & 0 & 0 & \frac{271}{60480} \\ 0 & 0 & 0 & 0 & 0 & -\frac{863}{60480} \end{pmatrix};$$

$$B_1 = \begin{pmatrix} \frac{2713}{2520} & -\frac{15487}{20160} & \frac{586}{945} & -\frac{6737}{20160} & \frac{263}{2520} & -\frac{863}{60480} \\ \frac{349}{840} & \frac{5221}{6720} & -\frac{254}{945} & \frac{811}{6720} & -\frac{29}{840} & \frac{271}{60480} \\ -\frac{23}{504} & \frac{10273}{20160} & \frac{586}{945} & -\frac{2257}{20160} & \frac{67}{2520} & -\frac{191}{60480} \\ \frac{67}{2520} & -\frac{2257}{20160} & \frac{586}{945} & \frac{10273}{20160} & -\frac{23}{504} & \frac{271}{60480} \\ -\frac{29}{840} & \frac{811}{6720} & -\frac{254}{945} & \frac{5221}{6720} & \frac{349}{840} & -\frac{863}{60480} \\ \frac{263}{2520} & -\frac{6737}{20160} & \frac{586}{945} & -\frac{15487}{20160} & \frac{2713}{2520} & \frac{19087}{60480} \end{pmatrix}$$

And

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$$A_0=\begin{pmatrix}1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\0&1&1&0&0&0&0\\0&0&1&1&0&0&0\\0&0&0&1&1&0&0\\0&0&0&0&1&1&0\\0&0&0&0&0&1&1\end{pmatrix};A_1=\begin{pmatrix}0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&1\end{pmatrix}$$

$$B_0=\begin{pmatrix}0&0&0&0&0&0&\frac{5257}{17280}\\0&0&0&0&0&0&-\frac{275}{24192}\\0&0&0&0&0&0&\frac{13}{4480}\\0&0&0&0&0&0&-\frac{191}{120960}\\0&0&0&0&0&0&\frac{191}{120960}\\0&0&0&0&0&0&-\frac{13}{4480}\\0&0&0&0&0&0&\frac{275}{24192}\end{pmatrix}$$

$$B_1=\begin{pmatrix}\frac{139849}{120960}&-\frac{4511}{4480}&\frac{123133}{120960}&-\frac{88547}{120960}&\frac{1537}{4480}&-\frac{11351}{120960}&\frac{275}{24192}\\\frac{5311}{13440}&\frac{11261}{13440}&-\frac{44797}{120960}&\frac{2987}{13440}&-\frac{1283}{13440}&\frac{2999}{120960}&-\frac{13}{4480}\\-\frac{4183}{120960}&\frac{6403}{13440}&\frac{9077}{13440}&-\frac{20227}{120960}&\frac{803}{13440}&-\frac{191}{13440}&\frac{191}{120960}\\\frac{1879}{120960}&-\frac{353}{4480}&\frac{68323}{120960}&\frac{68323}{120960}&-\frac{353}{4480}&\frac{1879}{120960}&-\frac{191}{120960}\\-\frac{191}{13440}&\frac{803}{13440}&-\frac{20227}{120960}&\frac{9077}{13440}&\frac{6403}{13440}&-\frac{4183}{120960}&\frac{13}{4480}\\\frac{2999}{120960}&-\frac{1283}{13440}&\frac{2987}{13440}&-\frac{44797}{120960}&\frac{11261}{13440}&\frac{5311}{13440}&-\frac{275}{24192}\\-\frac{11351}{120960}&\frac{1537}{4480}&-\frac{88547}{120960}&\frac{123133}{120960}&-\frac{4511}{4480}&\frac{139849}{120960}&\frac{5257}{17280}\end{pmatrix}$$

$$Y_m=\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\\y_{n+7}\end{pmatrix};Y_{m-1}=\begin{pmatrix}y_{n-6}\\y_{n-5}\\y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_{n}\end{pmatrix};F_m=\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\\f_{n+7}\end{pmatrix};F_{m-1}=\begin{pmatrix}f_{n-6}\\f_{n-5}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_{n}\end{pmatrix}$$

Substituting the test scalar equation $y'=\lambda y$ $(\lambda<0,\lambda$ is complex$)$ into (10) and taking $\lambda h=\bar{h}$ to get:

$$A_0Y_m=A_1Y_{m-1}+\bar{h}(B_0F_{m-1}+B_1F_m) \tag{11}$$

The stability polynomial of (8) and (9) are obtained by evaluating:

$$R(t;\bar{h})=Det\left[(A_0-\bar{h}B_1)t-(A_1+\bar{h}B_0)\right]=0 \tag{12}$$

to give respectively,

$$\left.\begin{aligned}R(t;\bar{h})&=\frac{1}{7}\bar{h}^6t^6-\frac{1}{7}\bar{h}^6t^5-\frac{393}{280}\bar{h}^5t^6+\frac{173}{168}\bar{h}^5t^5+\frac{21089}{3780}\bar{h}^4t^6-\frac{429731}{151200}\bar{h}^4t^5\\&-\frac{28583}{2520}\bar{h}^3t^6+\frac{1662797}{453600}\bar{h}^3t^5+\frac{15317}{1260}\bar{h}^2t^6-\frac{105523}{50400}\bar{h}^2t^5-\frac{11699}{1890}\bar{h}t^6+\frac{57}{160}\bar{h}t^5+t^6\end{aligned}\right\} \tag{13}$$

And

$$\left.\begin{aligned}R(t;\bar{h})&=-\frac{1}{8}\bar{h}^7t^7-\frac{1}{8}\bar{h}^7t^6+\frac{1037}{720}\bar{h}^6t^7+\frac{8551}{10080}\bar{h}^6t^6-\frac{2999}{432}\bar{h}^5t^7\\&-\frac{456437}{302400}\bar{h}^5t^6+\frac{7771}{432}\bar{h}^4t^7-\frac{1922993}{907200}\bar{h}^4t^6-\frac{80}{3}\bar{h}^3t^7+\frac{2893799}{259200}\bar{h}^3t^6+\frac{1187}{54}\bar{h}^2t^7\\&-\frac{245583959}{16329600}\bar{h}^2t^6-\frac{1177}{135}\bar{h}t^7+\frac{953479}{120960}\bar{h}t^6+t^7-t^6\end{aligned}\right\} \tag{14}$$

To establish the zero stability of the methods, we set $\bar{h}=0$ in (13) and (14) to get:

$$R(t;\bar{h})=t^6=0 \tag{15}$$

And

$$R(t;\bar{h})=t^7-t^6=0 \tag{16}$$

Solving (15) and (16) using Maple soft environment gives the following roots:

$t=0,\ t=0,\ t=0,\ t=0,\ t=0,t=0$

And

$t=1,\ t=0,\ t=0,\ t=0,\ t=0,t=0,t=0$ (17)

Hence, method (6) and (7) are zero stable by definition 1.

We plot the region of absolute stability of (6) and (7) which is determined by taking $t=e^{i\theta}$ into (13) and (14) respectively. The absolute stability graph is plotted using Matlab soft environment and is given in Figure 1 and 2.

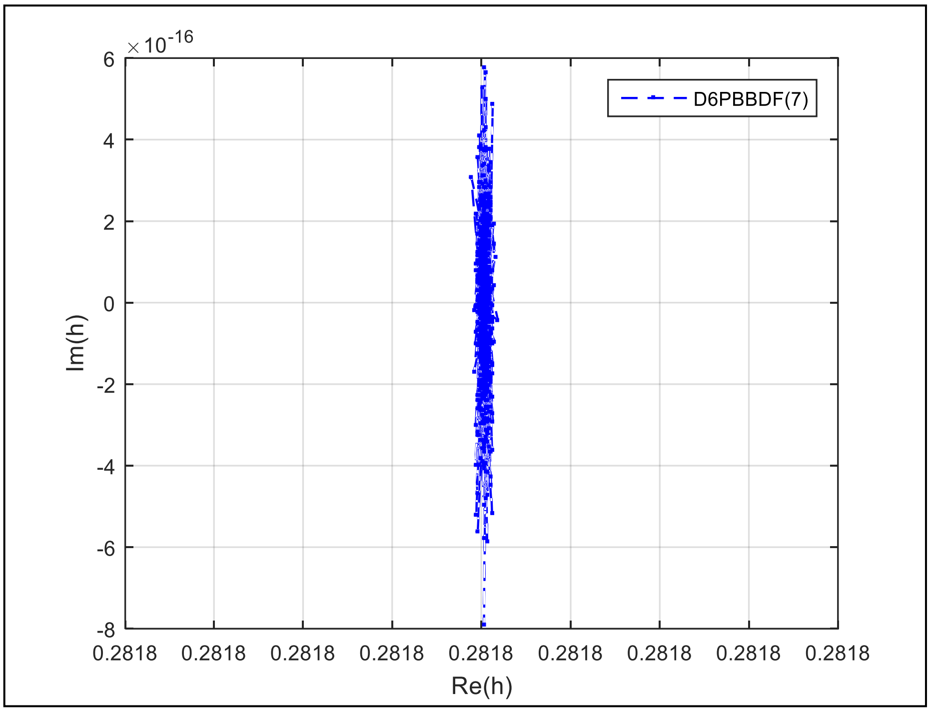

Fig 1:- Absolute stability region of D6PBBDF(7)

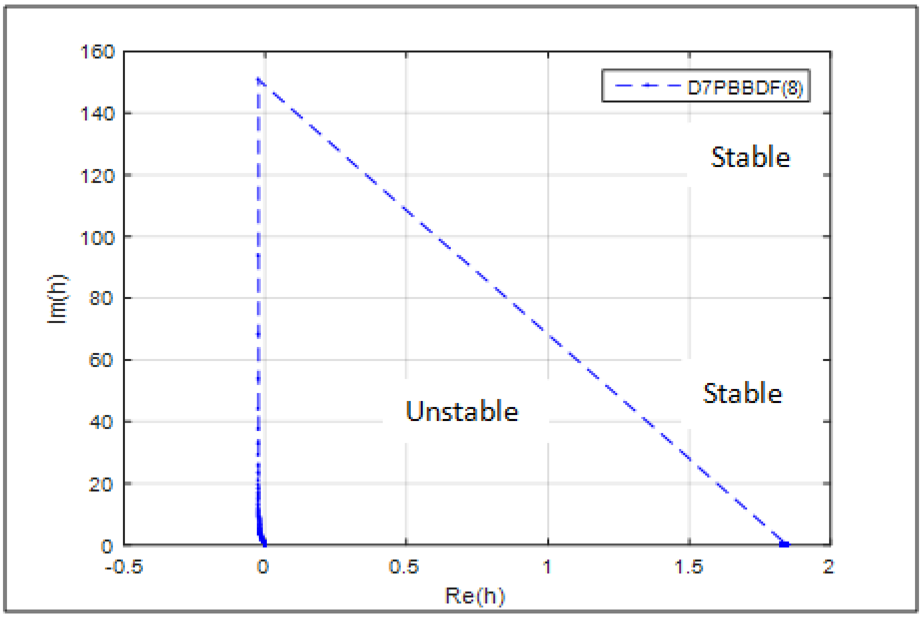

Fig 2:- Absolute stability region of D7PBBDF(8)

Figure 1 and 2 indicate that the entire (negative) left half complex plane represents the region of absolute stability for method (6) and (7).

- *Order and error constant*

We investigate the order and error constant of (6) and (7) using Maple soft environment to get:

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$$C_8 = \begin{pmatrix} \frac{275}{24192} \\ -\frac{13}{4480} \\ \frac{191}{120960} \\ -\frac{191}{120960} \\ \frac{13}{4480} \\ -\frac{275}{24192} \end{pmatrix}, C_9 = \begin{bmatrix} -\frac{33953}{3628800} \\ \frac{7297}{3628800} \\ -\frac{3233}{3628800} \\ \frac{2497}{3628800} \\ -\frac{3233}{3628800} \\ \frac{7297}{3628800} \\ -\frac{33953}{3628800} \end{bmatrix}$$

, implying that they are of order 7 and 8 respectively.

- *Convergence of the methods*

For any linear multistep method (LMM) to be convergent, it must be both zero stable and consistent. We shall discuss convergence of methods (6) and (7) as below.

Definition 3: Method (6) and (7) are consistent if and only if the following conditions are fulfilled:

The order $p \geq 1$

$$\sum_{j=0}^{3} D_j = 0,$$

$$\sum_{j=0}^{3} jD_j = \sum_{j=0}^{3} G_j$$

where, $D_{j's}$ and $G_{j's}$ are matrices.

- *Remark:*

Condition (18) is sufficient for the associated block methods to be consistent, i.e. $p \geq 1$ (Jator, 2007).

Thus, (6) and (7) are consistent since the order $p = 7, 8 > 1$. Since, the methods are both zero stable and consistent, they thus converge.

- *Implementation of the method*

The new methods are self-starting formulas. Hence, all approximate solutions are obtained simultaneously in block using Maple soft environment.

Definition 4: Let y_i and $y(x_i)$ be the approximate and exact solution of (1) respectively, then the maximum error is evaluated by using the formula:

$$\text{MAXE} = \max_{1 \leq t \leq NS} \left| (y_i)_t - (y(x_i))_t \right|$$

where, NS is the total number of steps.

- *Test Examples*

The following first order stiff initial value problems in (ODEs) are used to prove the accuracy of the method.

Example 1: [Nasir, *et al.*,(2015)]

$$y' = -1000y + 3000 - 2000e^{-x}, \ y(0) = 0, \ 0 \leq x \leq 1$$

$$\text{Exact solution: } y(x) = 3 - 0.998e^{-1000x} - 2.002e^{-x} \quad (18)$$

$$Eigenvalue: \ \lambda = -1000 \quad (19)$$

Example 2: [Nasir, *et al.*,(2015)]

$$y' = -100(y - \sin x) + \cos x, \ y(0) = 0, \ 0 \leq x \leq 1 \quad (20)$$

$$\text{Exact solution: } y(x) = \sin x$$

$$Eigenvalue: \ \lambda = -100$$

Example 3: [Mahayadin, Othman and Ibrahim, (2014)**]**

$$y' = -100(y - x^3) + 3x^2, \ y(0) = 0, \ 0 \leq x \leq 10$$

$$\text{Exact solution: } y(x) = x^3$$

$$\text{Eigenvalue: } \lambda = -100$$

Example 4: [Aksah, *et al.*,(2019)]

$$y' = -20y + 20\sin x + \cos x, y(0) = 1, 0 \leq x \leq 2$$

$$\text{Exact solution: } y(x) = \sin x + e^{-20x}$$

$$\text{Eigenvalue: } \lambda = -20$$

Example 5: [Babangida, Musa and Ibrahim, (2016)]

$$y_1' = -20y_1 - 19y_2, \ y_1(0) = 2, \ 0 \leq x \leq 20$$

$$y_2' = -19y_1 - 20y_2, \ y_2(0) = 0$$

$$\text{Exact solution: } y_1(x) = e^{-39x} + e^{-x}, \ y_2(x) = e^{-39x} - e^{-x}$$

$$\text{Eigenvalues: } \lambda = -1 \text{ and } -39$$

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III. NUMERICAL RESULTS

The tables below show the results from applying the new methods (6) and (7) with comparison to some existing numerical methods in terms of absolute maximum error. The following notations interpret the elements in the tables:

SDIBBDF : Singly diagonally implicit BBDF method by Aksah *et al.*, (2019)
BBDF(4) and BBDF(5) : Block backward differentiation formula of order four and five by Nasir *et al.*, (2015)
DIBBDF : Diagonally implicit BBDF method by Zawawi, (2014)
DI2BBDF : Diagonally implicit 2-Point BBDF method by Zawawi, *et al.*,(2012)
NDISBBDF : New Diagonally Implicit Super Class of Block Backward Differentiation Formula by Babangida, Musa, and Ibrahim (2016).
3PVSBBDF : 3-Point BBDF formulae with variable step size by Mahayadin *et al.*,(2014)
Odes15s : VSVO solver based on the numerical differentiation formulas (NDFs)
Ode23s : Modified Rosenbrock formula of order 2
NS : Number of steps taken
h : Stepsize
MAXE : Maximum error

Step-size (h)	**Method**	**NS**	**MAXE**
10^{-3}	BBDF(4) BBDF(5) D6PBBDF(7) D7PBBDF(8)	1000 1000 1000 1000	7.27898e+108 8.55887e+202 7.37700e-007 7.37700e-007
10^{-4}	BBDF(4) BBDF(5) D6PBBDF(7) D7PBBDF(8)	10000 10000 10000 10000	1.43379e-001 4.65939e-003 7.37700e-007 7.32700e-007

Table 1:- Numerical results for example 1

Step-size (h)	**Method**	**NS**	**MAXE**
10^{-2}	BBDF(4) BBDF(5) D6PBBDF(7) D7PBBDF(8)	100 100 100 100	8.28814e+006 5.18981e+013 4.00000e-010 6.00000e-010
10^{-3}	BBDF(4) BBDF(5) D6PBBDF(7) D7PBBDF(8)	1000 1000 1000 1000	1.55450e-003 1.67200e-005 1.00000e-009 1.60000e-009

Table 2:- Numerical results for Example 2

Step-size (h)	**Method**	**NS**	**MAXE**
10^{-2}	3PVSBBDF D6PBBDF(7) D7PBBDF(8)	- 1000 1000	2.07215e+080 1.40000e-009 1.00000e-009
10^{-3}	3PVSBBDF D6PBBDF(7) D7PBBDF(8)	- 10000 10000	1.79834e-003 4.80000e-008 4.80000e-008

Table 3:- Numerical results for Example 3

Step-size (h)	Method	NS	MAXE
10^{-2}	DIBBDF	-	9.19710e-002
	SDIBBDF	-	4.17749e-002
	Ode15s	-	8.36909e-003
	Ode23s	-	4.07991e-003
	D6PBBDF(7)	200	2.80000e-009
	D7PBBDF(8)	200	9.00000e-010
10^{-4}	DIBBDF	-	1.46293e-003
	SDIBBDF	-	4.94770e-006
	Ode15s	-	1.66322e-004
	Ode23s	-	1.83868e-004
	D6PBBDF(7)	10000	2.53390e-008
	D7PBBDF(8)	10000	1.07439e-007

Table 4:- Numerical results for Example 4

Step-size (h)	Method	NS	MAXE
10^{-2}	DI2BBDF	1000	6.85453e-002
	NDISBBDF	1000	7.15278e-002
	D6PBBDF(7)	1000	6.03000e-012
	D7PBBDF(8)	1000	1.92000e-012
10^{-3}	DI2BBDF	1000	2.60436e-002
	NDISBBDF	1000	2.32062e-003
	D6PBBDF(7)	1000	3.79000e-008
	D7PBBDF(8)	1000	4.29000e-008

Table 5:- Numerical results for Example 5

IV. DISCUSSION

Table 1 show that at step-size, $h=10^{-3}$, the new methods D6PBBDF(7) and D7PBBDF(8) have absolute maximum error of 7.37700e-007 each with BBDF(4) and BBDF(5) have 7.27898e+108 and 8.55887e+202. When the step-size is $h=10^{-4}$, D6PBBDF(7) and D7PBBDF(8) have 7.37700e-007 and 7.32700e-007 , with D7PBBDF(8) showing improvement, BBDF(4) and BBDF(5) have 1.43379e-001 and 4.65939e-003. Thus, Table 1 shows that D7PBBDF(8) outperformed D6PBBDF(7), BBDF(4) and BBDF(5), though, there was no improvement as step-sizes decreases. Also, in Table 2, when $h=10^{-2}$, the new methods, D6PBBDF(7) and D7PBBDF(8) have 4.00000e-010 and 6.00000e-010, BBDF(4) and BBDF(5) have 8.28814e+006 and 5.18981e+013, with $h=10^{-3}$, D6PBBDF(7) and D7PBBDF(8) have 1.00000e-009 and 1.60000e-009, BBDF(4) and BBDF(5) have 1.55450e-003 and 1.67200e-005. Thus, in Table 2, D6PBBDF(7) outperformed D7PBBDF(8), BBDF(4) and BBDF(5) respectively. In Table 3, when $h=10^{-2}$, 3PVSBBDF has 2.07215e+080, the new methods, D6PBBDF(7) and D7PBBDF(8) have 1.40000e-009 and 1.00000e-009, when $h=10^{-3}$, 3PVSBBDF has 1.79834e-003, D6PBBDF(7) and D7PBBDF(8) have 4.8000e-008 and 4.80000e-008. Table 3, generally indicates that D7PBBDF(8) is better preferred to D6PBBDF(7) for problem 3 as it shows better performance even as step-size reduces. In Table 4, when $h=10^{-2}$, DIBBDF, SDIBBDF, Ode15s, Ode23s, D6PBBDF(7) and D7PBBDF(8) have 9.19710e-002, 4.17749e-002, 8.36909e-003, 4.07991e-003, 2.80000e-009 and 9.00000e-010 respectively. Also, with $h=10^{-4}$, DIBBDF, SDIBBDF, Ode15s, Ode23s, D6PBBDF(7) and D7PBBDF(8) have 1.46293e-003, 4.94770e-006, 1.66322e-004, 1.83868e-004, 2.53390e-008 and 1.07439e-007. Generally, Table 4, showed that the two new methods D6PBBDF(7) and D7PBBDF(8) did not show considerable improvement but at $h=10^{-2}$, D7PBBDF(8) has a small scale error when compared to D6PBBDF(7), DIBBDF, SDIBBDF, Ode15s, Ode23s respectively while when $h=10^{-4}$, D6PBBDF(7) is better in performance when compared to D7PBBDF(8), DIBBDF, SDIBBDF, Ode15s, Ode23s respectively. Similarly, from Table 5, it can be seen that with $h=10^{-2}$, DI2BBDF has absolute maximum error of 6.85453e-002, NDISBBDF has 7.15278e-002, D6PBBDF(7) has 6.03000e-012 and D7PBBDF(8) has 1.92000e-12. Also, with $h=10^{-3}$, DI2BBDF has absolute maximum error of 2.620436e-002, NDISBBDF has 2.32062e-003, D6PBBDF(7) has 3.79000e-008 and D7PBBDF(8) has 4.29000e-008. Thus, Table 5 generally indicated that though the new methods (D6PBBDF(7) and D7PBBDF(8)) outperformed DI2BBDF and NDISBBDF respectively, but did not tend to show improvement in terms of absolute maximum error as step-sizes tend to zero.

However, the maximum error implied that the approximate solutions tend to the exact solutions as the iteration processes continue. Hence, the new method converges faster than the existing methods on the respective problems considered, though, no considerable improvements exist on some of the problems considered as the step-sizes reduce, but improvement is promising on problem 1, especially, with D7PBBDF(8).

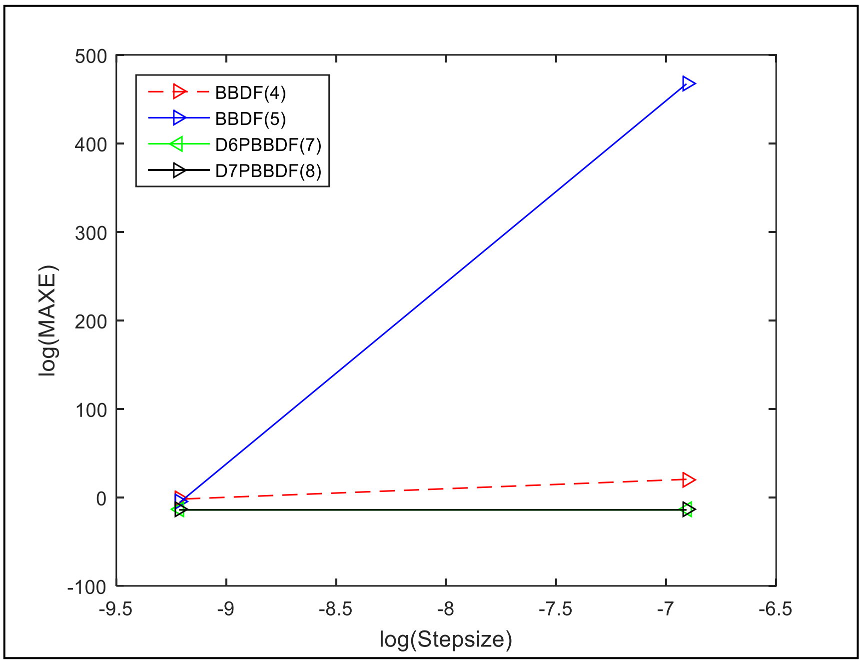

Fig 3:- Comparison of efficiency curves in terms of error for Example 1

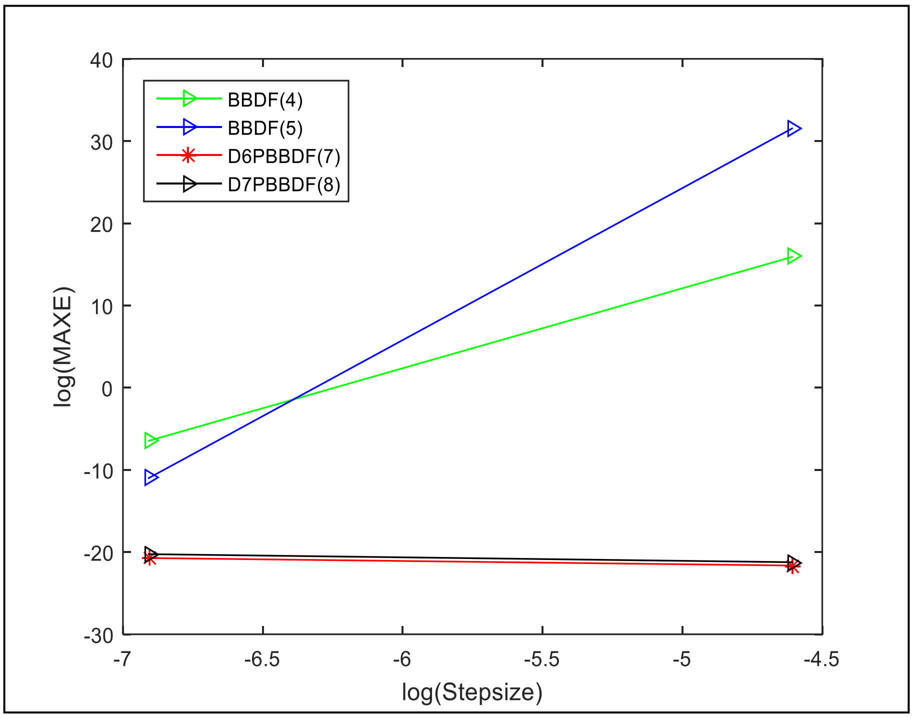

Fig 4:- Comparison of efficiency curves in terms of error for problem 2

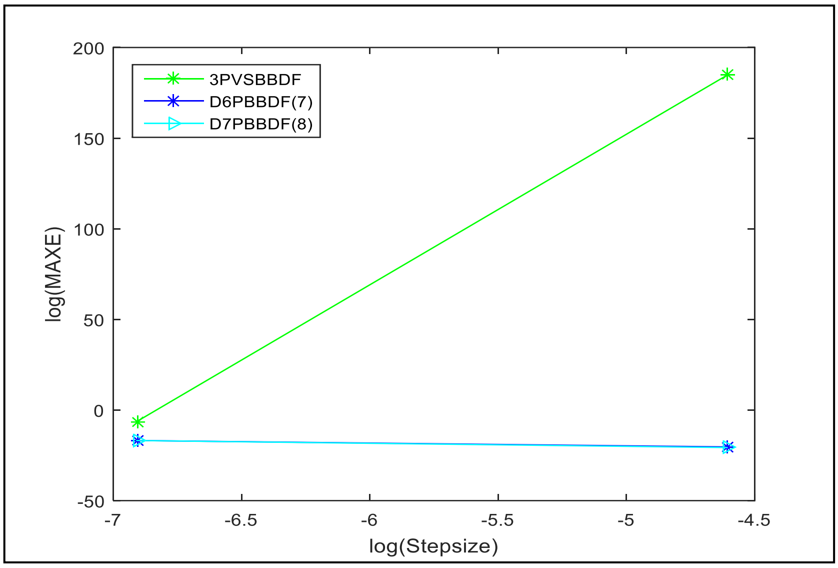

Fig 5:- Comparison of efficiency curves in terms of error for problem 3

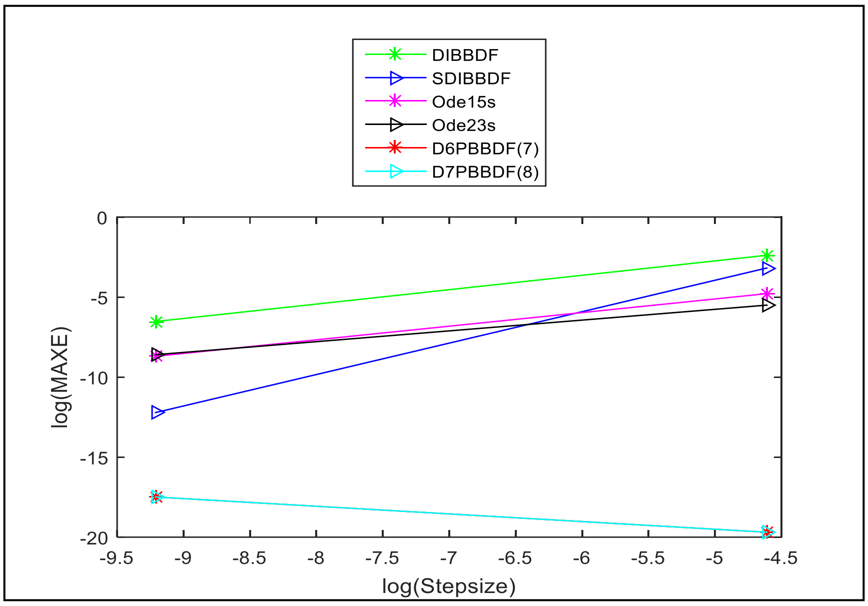

Fig 6:- Comparison of efficiency curves in terms of error for problem 4

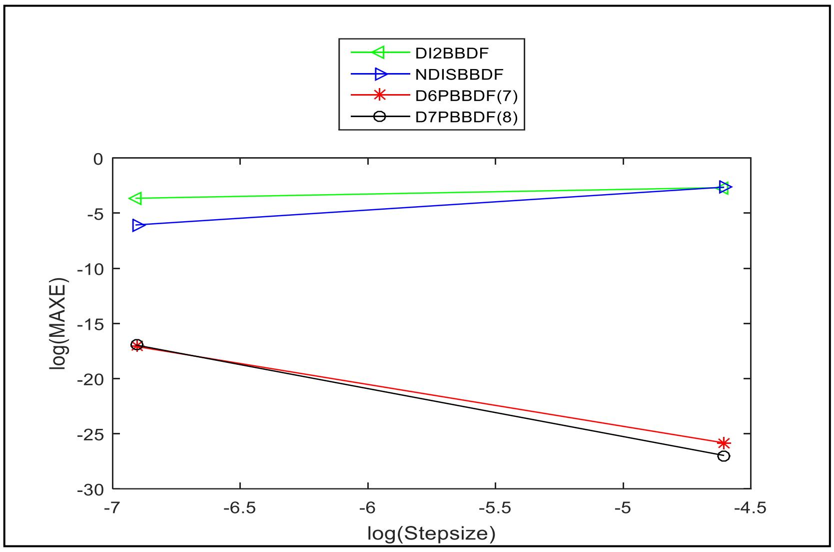

Fig 7:- Comparison of efficiency curves in terms of error for problem 5

Figure 3-7 indicate clearly the comparison of the new methods with some of the existing methods using efficiency curves. The curves imply that as the step-size reduces scale errors become smaller. This is evident specifically in problem 1where D6PBBDF(7) and D7PBBDF(8) improved and the compared methods. Figure 4-7 though did not show improvement considering examples 2-5 in the new methods but the new methods showed improved accuracy than some of the existing methods considered. This behaviour could possibly be as a result of the stiff nature of the problems considered or otherwise. Thus, step-size restriction likely to be bound on the methods since they did not show improvement as step-sizes tend to zero on some of the problems considered.

V. CONCLUSION

A new uniform higher 6,7-point block methods have been developed through interpolation and collocation approaches using power series expansion as the approximate solution. It has been established that they are of order 7 and 8 respectively. The region of absolute stability showed that the methods are A-stable. The efficiency of the methods on test problems showed that the accuracy of the new methods, D6PBBDF(7) and D7PBBDF(8) are better off in terms of absolute maximum error when compared to BBDF(4), BBDF(5), DIBBDF, SDIBBDF, Ode15s, Ode23s, DI2BBDF and NDISBBDF respectively. Hence, new methods for solving first order stiff initial value problems in ordinary differential equations (ODEs) have been developed.

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