

Some Definite Integral Associated to Struve and Modified Struve Function in the form of Hypergeometric Function

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Abstract:- In this paper we have derived some definite integral involving Struve and modified struve Function in association with Hypergeometric function.

Keywords :- Bessel Function, Hypergeometric Function, Struve function.

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I. INTRODUCTION

Yurry A. Brychkov [Brychkov p.221(4.11.5.2-4.11.5.5)] has derived the following formulae

$$\int_0^1 c \cos^{-1} x H_0(ax) dx = \frac{1}{a} [C - Ci(a) + \log a], \quad [|\arg a < \pi|]. \quad (1.1)$$

$$\int_0^1 \frac{1}{x} \cos^{-1} x H_1(ax) dx = C - 1 - Ci(a) + \log a + \frac{1}{a} \sin a, \quad [|\arg a < \pi|]. \quad (1.2)$$

$$\int_0^1 c \cos^{-1} x L_0(ax) dx = \frac{1}{a} [Chi(a) - C - \log a], \quad [|\arg a < \pi|]. \quad (1.3)$$

$$\int_0^1 \frac{1}{x} \cos^{-1} x L_1(ax) dx = 1 - C + Chi(a) - \log a - \frac{1}{a} \sinh a, \quad [|\arg a < \pi|]. \quad (1.4)$$

Struve functions are solutions of the non-homogeneous Bessel's differential equation:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - a^2)y = \frac{4(\frac{x}{2})^{\alpha+1}}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \quad (1.5)$$

and are defined as:

$$H_\alpha(x) = \frac{2(\frac{x}{2})^\alpha}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} \sin(x \cos \theta) \sin^{2\alpha}(\theta) d\theta \quad (1.6)$$

Modified Struve function is:

$$L_\alpha(x) = I_{-\alpha}(x) - \frac{2\left(\frac{x}{2}\right)^\alpha}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \int_0^\infty s \sin(xu)(1 + u^2)^{\alpha - \frac{1}{2}} du \tag{1.7}$$

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel’s differential equation that are finite at the origin ($x = 0$) for integer or positive α , and diverge as x approaches zero for negative non-integer α (See[13]). It is possible to define the function by its Taylor series expansion around $x = 0$.

$$J_\alpha(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.8}$$

where $\Gamma(z)$ is the gamma function, a shifted generalization of the factorial function to non-integer values. The Bessel function of the first kind is an entire function if α is an integer.

The Bessel functions are valid even for complex arguments x , and an important special case is that of a purely imaginary argument(See[13]). In this case, the solutions to the Bessel equation are called the modified Bessel functions(or occasionally the hyperbolic Bessel functions) of the first and second kind. The first kind of modified Bessel function is defined as

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^\infty \frac{1}{m! \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.9}$$

A generalized hypergeometric function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k + a_1)(k + a_2) \dots (k + a_p)}{(k + b_1)(k + b_2) \dots (k + b_q)(k + 1)} z. \tag{1.10}$$

Where $k + 1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p & ; \\ b_1, b_2, \dots, b_q & ; \end{matrix} \middle| z \right] = \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \dots (a_p)_k z^k}{(b_1)_k (b_2)_k \dots (b_q)_k k!} \tag{1.11}$$

where the parameters b_1, b_2, \dots, b_q are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q + 1$, diverges for all $z, z \neq 0$ if $p > q + 1$ [Luke p.156(3)].

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss’s hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial, falling sequential product, or lower factorial) is defined as the polynomial[Steffensen p.8]

$$\begin{aligned}
 (x)_n &= x(x-1)(x-2)\dots(x-n+1) = \prod_{k=1}^n (x-k+1) \\
 &= \prod_{k=0}^{n-1} (x-k)
 \end{aligned}
 \tag{1.12}$$

II. MAIN FORMULAE OF THE INTEGRATION

$$\begin{aligned}
 \int_0^1 c \cos^{-1} x H_5(ax) dx & \\
 &= \frac{1}{2546775\pi} [32a^6 {}_3F_4(1, \frac{7}{2}, 4; \frac{3}{2}, \frac{9}{2}, \frac{9}{2}, \frac{13}{2}; -\frac{a^2}{4})].
 \end{aligned}
 \tag{2.1}$$

$$\int_0^1 c \cos^{-1} x L_5(ax) dx = \frac{1}{2546775\pi} [32a^6 {}_3F_4(1, \frac{7}{2}, 4; \frac{3}{2}, \frac{9}{2}, \frac{9}{2}, \frac{13}{2}; \frac{a^2}{4})]. \tag{2.2}$$

$$\begin{aligned}
 \int_0^1 x^2 \cos^{-1} x H_5(ax) dx & \\
 &= \frac{1}{29469825\pi} [256a^6 {}_3F_4(1, \frac{9}{2}, 5; \frac{3}{2}, \frac{11}{2}, \frac{11}{2}, \frac{13}{2}; -\frac{a^2}{4})].
 \end{aligned}
 \tag{2.3}$$

$$\begin{aligned}
 \int_0^1 x^2 \cos^{-1} x L_5(ax) dx & \\
 &= \frac{1}{29469825\pi} [256a^6 {}_3F_4(1, \frac{9}{2}, 5; \frac{3}{2}, \frac{11}{2}, \frac{11}{2}, \frac{13}{2}; \frac{a^2}{4})].
 \end{aligned}
 \tag{2.4}$$

$$\begin{aligned}
 \int_0^1 x^4 \cos^{-1} x H_5(ax) dx & \\
 &= \frac{1}{79241085\pi} [512a^6 {}_3F_4(1, \frac{11}{2}, 6; \frac{3}{2}, \frac{13}{2}, \frac{13}{2}, \frac{13}{2}; -\frac{a^2}{4})].
 \end{aligned}
 \tag{2.5}$$

$$\begin{aligned}
 \int_0^1 x^4 \cos^{-1} x L_5(ax) dx & \\
 &= \frac{1}{79241085\pi} [512a^6 {}_3F_4(1, \frac{11}{2}, 6; \frac{3}{2}, \frac{13}{2}, \frac{13}{2}, \frac{13}{2}; \frac{a^2}{4})].
 \end{aligned}
 \tag{2.6}$$

$$\begin{aligned}
 \int_0^1 x^6 \cos^{-1} x H_5(ax) dx & \\
 &= \frac{1}{405810405\pi} [2048a^6 {}_2F_3(1, 7; \frac{3}{2}, \frac{15}{2}, \frac{15}{2}; -\frac{a^2}{4})].
 \end{aligned}
 \tag{2.7}$$

$$\begin{aligned}
 \int_0^1 x^6 \cos^{-1} x L_5(ax) dx & \\
 &= \frac{1}{405810405\pi} [2048a^6 {}_2F_3(1, 7; \frac{3}{2}, \frac{15}{2}, \frac{15}{2}; \frac{a^2}{4})].
 \end{aligned}
 \tag{2.8}$$

$$\int_0^1 x^8 \cos^{-1}x H_5(ax)dx = \frac{1}{1003377375\pi} [4096a^6 {}_3F_4(1, \frac{15}{2}, 8; \frac{3}{2}, \frac{13}{2}, \frac{17}{2}, \frac{17}{2}; -\frac{a^2}{4})]. \tag{2.9}$$

$$\int_0^1 x^8 \cos^{-1}x L_5(ax)dx = \frac{1}{1003377375\pi} [4096a^6 {}_3F_4(1, \frac{15}{2}, 8; \frac{3}{2}, \frac{13}{2}, \frac{17}{2}, \frac{17}{2}; \frac{a^2}{4})]. \tag{2.10}$$

$$\int_0^1 x^{10} \cos^{-1}x H_5(ax)dx = \frac{1}{19331737425\pi} [65536a^6 {}_3F_4(1, \frac{17}{2}, 9; \frac{3}{2}, \frac{13}{2}, \frac{19}{2}, \frac{19}{2}; -\frac{a^2}{4})]. \tag{2.11}$$

$$\int_0^1 x^{10} \cos^{-1}x L_5(ax)dx = \frac{1}{19331737425\pi} [65536a^6 {}_3F_4(1, \frac{17}{2}, 9; \frac{3}{2}, \frac{13}{2}, \frac{19}{2}, \frac{19}{2}; \frac{a^2}{4})]. \tag{2.12}$$

$$\int_0^1 x^{12} \cos^{-1}x H_5(ax)dx = \frac{1}{45612792225\pi} [131072a^6 {}_3F_4(1, \frac{19}{2}, 10; \frac{3}{2}, \frac{13}{2}, \frac{21}{2}, \frac{21}{2}; -\frac{a^2}{4})]. \tag{2.13}$$

$$\int_0^1 x^{12} \cos^{-1}x L_5(ax)dx = \frac{1}{45612792225\pi} [131072a^6 {}_3F_4(1, \frac{19}{2}, 10; \frac{3}{2}, \frac{13}{2}, \frac{21}{2}, \frac{21}{2}; \frac{a^2}{4})]. \tag{2.14}$$

$$\int_0^1 x^{14} \cos^{-1}x H_5(ax)dx = \frac{1}{211739382855\pi} [524288a^6 {}_3F_4(1, \frac{21}{2}, 11; \frac{3}{2}, \frac{13}{2}, \frac{23}{2}, \frac{23}{2}; -\frac{a^2}{4})]. \tag{2.15}$$

$$\int_0^1 x^{14} \cos^{-1}x L_5(ax)dx = \frac{1}{211739382855\pi} [524288a^6 {}_3F_4(1, \frac{21}{2}, 11; \frac{3}{2}, \frac{13}{2}, \frac{23}{2}, \frac{23}{2}; \frac{a^2}{4})]. \tag{2.16}$$

$$\int_0^1 x^{16} \cos^{-1}x H_5(ax)dx = \frac{1}{484892352945\pi} [1048576a^6 {}_3F_4(1, \frac{23}{2}, 12; \frac{3}{2}, \frac{13}{2}, \frac{25}{2}, \frac{25}{2}; -\frac{a^2}{4})]. \tag{2.17}$$

$$\int_0^1 x^{16} \cos^{-1}x L_5(ax)dx = \frac{1}{484892352945\pi} [1048576a^6 \quad {}_3F_4(1, \frac{23}{2}, 12; \frac{3}{2}, \frac{13}{2}, \frac{25}{2}, \frac{25}{2}; \frac{a^2}{4})]. \tag{2.18}$$

$$\int_0^1 x^{18} \cos^{-1}x H_5(ax)dx = \frac{1}{4392140878125\pi} [8388608a^6 \quad {}_3F_4(1, \frac{25}{2}, 13; \frac{3}{2}, \frac{13}{2}, \frac{27}{2}, \frac{27}{2}; -\frac{a^2}{4})]. \tag{2.19}$$

$$\int_0^1 x^{18} \cos^{-1}x L_5(ax)dx = \frac{1}{4392140878125\pi} [8388608a^6 \quad {}_3F_4(1, \frac{25}{2}, 13; \frac{3}{2}, \frac{13}{2}, \frac{27}{2}, \frac{27}{2}; \frac{a^2}{4})]. \tag{2.20}$$

$$\int_0^1 \frac{1}{x^2} \cos^{-1}x H_5(ax)dx = \frac{1}{779625\pi} [16a^6 \quad {}_3F_4(1, \frac{5}{2}, 3; \frac{3}{2}, \frac{7}{2}, \frac{7}{2}, \frac{13}{2}; -\frac{a^2}{4})]. \tag{2.21}$$

$$\int_0^1 \frac{1}{x^2} \cos^{-1}x L_5(ax)dx = \frac{1}{779625\pi} [16a^6 \quad {}_3F_4(1, \frac{5}{2}, 3; \frac{3}{2}, \frac{7}{2}, \frac{7}{2}, \frac{13}{2}; \frac{a^2}{4})]. \tag{2.22}$$

$$\int_0^1 \frac{1}{x^4} \cos^{-1}x H_5(ax)dx = \frac{1}{93555\pi} [4a^6 \quad {}_2F_3(1, 2; \frac{5}{2}, \frac{5}{2}, \frac{13}{2}; -\frac{a^2}{4})]. \tag{2.23}$$

$$\int_0^1 \frac{1}{x^4} \cos^{-1}x L_5(ax)dx = \frac{1}{93555\pi} [4a^6 \quad {}_2F_3(1, 2; \frac{5}{2}, \frac{5}{2}, \frac{13}{2}; \frac{a^2}{4})]. \tag{2.24}$$

$$\int_0^1 \frac{1}{x^6} \cos^{-1}x H_5(ax)dx = \frac{1}{10395\pi} [2a^6 \quad {}_3F_4(1, \frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{13}{2}; -\frac{a^2}{4})]. \tag{2.25}$$

$$\int_0^1 \frac{1}{x^6} \cos^{-1}x L_5(ax)dx = \frac{1}{10395\pi} [2a^6 \quad {}_3F_4(1, \frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{13}{2}; \frac{a^2}{4})]. \tag{2.26}$$

$$\int_0^1 x \cos^{-1}x H_6(ax)dx = \frac{1}{383107725\pi} [256a^7 \quad {}_3F_4(1, \frac{9}{2}, 5; \frac{3}{2}, \frac{13}{2}, \frac{13}{2}, \frac{15}{2}; -\frac{a^2}{4})]. \tag{2.27}$$

$$\int_0^1 x \cos^{-1}x L_6(ax)dx = \frac{1}{383107725\pi} [256a^7 \quad {}_3F_4(1, \frac{9}{2}, 5; \frac{3}{2}, \frac{13}{2}, \frac{13}{2}, \frac{15}{2}; \frac{a^2}{4})]. \tag{2.28}$$

$$\int_0^1 x^3 \cos^{-1}x H_6(ax)dx = \frac{1}{1030134105\pi} [512a^7 {}_3F_4(1, \frac{11}{2}, 6; \frac{3}{2}, \frac{11}{2}, \frac{11}{2}, \frac{15}{2}; -\frac{a^2}{4})]. \tag{2.29}$$

$$\int_0^1 x^3 \cos^{-1}x L_6(ax)dx = \frac{1}{1030134105\pi} [512a^7 {}_3F_4(1, \frac{11}{2}, 6; \frac{3}{2}, \frac{11}{2}, \frac{11}{2}, \frac{15}{2}; \frac{a^2}{4})]. \tag{2.30}$$

$$\int_0^1 x^5 \cos^{-1}x H_6(ax)dx = \frac{1}{5275535265\pi} [2048a^7 {}_3F_4(1, \frac{13}{2}, 7; \frac{3}{2}, \frac{15}{2}, \frac{15}{2}, \frac{15}{2}; -\frac{a^2}{4})]. \tag{2.31}$$

$$\int_0^1 x^5 \cos^{-1}x L_6(ax)dx = \frac{1}{5275535265\pi} [2048a^7 {}_3F_4(1, \frac{13}{2}, 7; \frac{3}{2}, \frac{15}{2}, \frac{15}{2}, \frac{15}{2}; \frac{a^2}{4})]. \tag{2.32}$$

$$\int_0^1 x^7 \cos^{-1}x H_6(ax)dx = \frac{1}{13043905875\pi} [4096a^7 {}_2F_3(1, 8; \frac{3}{2}, \frac{17}{2}, \frac{17}{2}; -\frac{a^2}{4})]. \tag{2.33}$$

$$\int_0^1 x^7 \cos^{-1}x L_6(ax)dx = \frac{1}{13043905875\pi} [4096a^7 {}_2F_3(1, 8; \frac{3}{2}, \frac{17}{2}, \frac{17}{2}; \frac{a^2}{4})]. \tag{2.34}$$

$$\int_0^1 x^9 \cos^{-1}x H_6(ax)dx = \frac{1}{251312586525\pi} [65536a^7 {}_3F_4(1, \frac{17}{2}, 9; \frac{3}{2}, \frac{15}{2}, \frac{19}{2}, \frac{19}{2}; -\frac{a^2}{4})]. \tag{2.35}$$

$$\int_0^1 x^9 \cos^{-1}x L_6(ax)dx = \frac{1}{251312586525\pi} [65536a^7 {}_3F_4(1, \frac{17}{2}, 9; \frac{3}{2}, \frac{15}{2}, \frac{19}{2}, \frac{19}{2}; \frac{a^2}{4})]. \tag{2.36}$$

$$\int_0^1 x^{11} \cos^{-1}x H_6(ax)dx = \frac{1}{592966298925\pi} [131072a^7 {}_3F_4(1, \frac{19}{2}, 10; \frac{3}{2}, \frac{15}{2}, \frac{21}{2}, \frac{21}{2}; -\frac{a^2}{4})]. \tag{2.37}$$

$$\int_0^1 x^{11} \cos^{-1}x L_6(ax)dx = \frac{1}{592966298925\pi} [131072a^7 \quad {}_3F_4(1, \frac{19}{2}, 10; \frac{3}{2}, \frac{15}{2}, \frac{21}{2}, \frac{21}{2}; \frac{a^2}{4})]. \tag{2.38}$$

$$\int_0^1 x^{13} \cos^{-1}x H_6(ax)dx = \frac{1}{2752611977115\pi} [524288a^7 \quad {}_3F_4(1, \frac{21}{2}, 10; \frac{3}{2}, \frac{15}{2}, \frac{23}{2}, \frac{23}{2}; -\frac{a^2}{4})]. \tag{2.39}$$

$$\int_0^1 x^{13} \cos^{-1}x L_6(ax)dx = \frac{1}{2752611977115\pi} [524288a^7 \quad {}_3F_4(1, \frac{21}{2}, 10; \frac{3}{2}, \frac{15}{2}, \frac{23}{2}, \frac{23}{2}; \frac{a^2}{4})]. \tag{2.40}$$

$$\int_0^1 \frac{1}{x} \cos^{-1}x H_6(ax)dx = \frac{1}{33108075\pi} [32a^7 \quad {}_3F_4(1, \frac{7}{2}, 4; \frac{3}{2}, \frac{9}{2}, \frac{9}{2}, \frac{15}{2}; -\frac{a^2}{4})]. \tag{2.41}$$

$$\int_0^1 \frac{1}{x} \cos^{-1}x L_6(ax)dx = \frac{1}{33108075\pi} [32a^7 \quad {}_3F_4(1, \frac{7}{2}, 4; \frac{3}{2}, \frac{9}{2}, \frac{9}{2}, \frac{15}{2}; \frac{a^2}{4})]. \tag{2.42}$$

$$\int_0^1 \frac{1}{x^3} \cos^{-1}x H_6(ax)dx = \frac{1}{10135125\pi} [16a^7 \quad {}_3F_4(1, \frac{5}{2}, 3; \frac{3}{2}, \frac{7}{2}, \frac{7}{2}, \frac{15}{2}; -\frac{a^2}{4})]. \tag{2.43}$$

$$\int_0^1 \frac{1}{x^3} \cos^{-1}x L_6(ax)dx = \frac{1}{10135125\pi} [16a^7 \quad {}_3F_4(1, \frac{5}{2}, 3; \frac{3}{2}, \frac{7}{2}, \frac{7}{2}, \frac{15}{2}; \frac{a^2}{4})]. \tag{2.43}$$

$$\int_0^1 \frac{1}{x^5} \cos^{-1}x H_6(ax)dx = \frac{1}{1216215\pi} [4a^7 \quad {}_2F_3(1, 2; \frac{5}{2}, \frac{5}{2}, \frac{15}{2}; -\frac{a^2}{4})]. \tag{2.44}$$

$$\int_0^1 \frac{1}{x^5} \cos^{-1}x L_6(ax)dx = \frac{1}{1216215\pi} [4a^7 \quad {}_2F_3(1, 2; \frac{5}{2}, \frac{5}{2}, \frac{15}{2}; \frac{a^2}{4})]. \tag{2.45}$$

$$\int_0^1 \frac{1}{x^7} \cos^{-1}x H_6(ax)dx = \frac{1}{135135\pi} [2a^7 \quad {}_3F_4(1, \frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{15}{2}; -\frac{a^2}{4})]. \tag{2.46}$$

$$\int_0^1 \frac{1}{x^7} \cos^{-1} x L_6(ax) dx = \frac{1}{135135\pi} [2a^7 {}_3F_4(1, \frac{1}{2}, 1; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{15}{2}; \frac{a^2}{4})]. \quad (2.47)$$

III. DERIVATION OF INTEGRATION

By applying computational technique one can derive the Integration.

IV. CONCLUSION

In our effort we have developed some definite integrals. But, one can develop such type of integrals which are very useful for different area of science and technology by using these integrals. So our hope that the development granted in this research will encourage in this important area of special functions.

REFERENCES

- [1]. Abramowitz, Milton., A and Stegun, Irene ; *Handbook of Mathematical Functions with Formulas , Graphs , and Mathematical Tables*, National Bureau of Standards, 1970.
- [2]. Aomoto, Kazuhiko, Kita, Michitake ; *Theory of Hypergeometric functions*, Springer Tokyo Dordrecht Heidelberg London New York, 2011.
- [3]. Brychkov, Y.A.; *Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas*. CRC Press, Taylor & Francis Group, London, U.K, 2008.
- [4]. Bowman, F.; *Introduction to Bessel functions*. Dover Publications Inc., New York, 1958.
- [5]. Gasper, G, Rahman, Mizan ; *Basic Hypergeometric Series*, Cambridge University Press, Cambridge CB2 2RU, UK, 2004.
- [6]. Gauss, C. F. ; *Disquisitiones generales circa seriem infinitam ...* , Comm. soc. reg. sci. Gott. rec., 2(1813), 123-162.
- [7]. Izharul H Khan and Salahuddin ; Certain Definite Integral Involving Struve and Modified Struve Function in the form of Hypergeometric Function, *International journal of Mathematics Trends and Technology*, 66(8)(2020), 92-99.
- [8]. Koepf, W.; *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities*. Braunschweig, Germany: Vieweg, 1998.
- [9]. Luke, Y. L.; *Mathematical functions and their approximations*. Academic Press Inc., London., 1975.
- [10]. Mathai, A.M., Haubold, H.J. ; *Special Functions for Applied Scientists*, Springer , New York, 2008.
- [11]. P. Appell, Sur une formule de M. Tisserand et sur les fonctions hypergéométriques de deux variables, *J. Math. Pures Appl.*, (3) 10 (1884) 407-428.
- [12]. Prudnikov, A.P., Brychkov, Yu. A. and Marichev, O.I.; *Integral and Series Vol 3: More Special Functions*, Nauka, Moscow, 2003.
- [13]. Salahuddin, Husain, Intazar; Certain New Formulae Involving Modified Bessel Function of First Kind, *Global Journal of Science Frontier Research (F)*, 13(10)(2013), 13-19.
- [14]. Salahuddin, Kholi, R. K.; New hypergeometric summation formulae arising from the summation formulae of Prudnikov, *South Asian Journal of Mathematics*, 4(2014), 192-196.
- [15]. Steffensen, J. F.; *Interpolation (2nd ed.)*, Dover Publications, U.S.A., 2006.