Some Inequalities Related to Riemann-Liouville Fractional Inequalities in Q-Calculus

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Abstract:- The purpose of this paper is to present different fractional order derivatives and inequalities that are commonly used in the literature, especially in Riemann-Liouville sense of Fractional inequalities in the context of q-calculus. In this current work we have presented some new inequalities related to the Riemann-Liouville Fractional inequalities in the context of q-calculus. Fractional calculus explores integrals and derivatives of functions involving non-integer order(s). Its application to Quantum calculus (q-Calculus), on the other hand focuses on investigations related to calculus without limits.

In recent times, these aspect of mathematics has attracted the attention of many researchers due to its high demand for solving complex systems in nature with anomalous dynamics. We therefore, introduce some new inequalities related to Riemann-Liouville fractional integral inequalities with limist via q-Calculus

***Keywords**:- Riemann-Liouville, Fractional calculus, q-Calculus, limits 2010.*

I. INTRODUCTION

Recently, fractional inequality has been used to enhance understanding in solving and describing various problems in Mathematics, especially rational differential equations and inequalities. It is worth stating the that since the integer-order integrals and derivatives, do not always apply adequately in many cases, in the current mathematical models, rational order derivatives come with reasons of fixing the identified gap.

Fractional calculus explores integrals and derivatives of functions which involve non-integer order(s). Fractional calculus is seen as the branch of Mathematics which generalizes the integer-order differentiation and integration to derivatives and integrals of arbitrary order. So many extensions have been attemped on some important formulae using fractional calculus. Researcher in these category include; Kilbas (2001),Kilbas et al. (2006) Annaby and Mansour (2012),Oldham and Spanier. (1974), Usta et al. (2017),Yanga (2015), Atangana and Secer (2013), and the reference therein.

It is very important to note that all the fractional derivative order definitions have their pros and cons. We therefore include Caputo derivatives for the purposes of comparison. We will then employ Quantum calculus (q-Calculus) on the in our effort to refocus on investigations related to 'calculus without limits'.

II. PRELIMINARIES

In this section we seek to explore some basic definitions as well as formulas which are necessary in this current work. There exists a vast literature on different definitions of fractional derivatives. The most popular ones are the RiemannLiouville and the Caputo derivatives. For Caputo we have

$$ {}_0^c\mathbb{D}_x^\alpha(f(x)) = \frac{1}{\Gamma(n-1)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^nf(t)}{dt^n}dt, \qquad (1)$$

for $n-1<\alpha\leq$.

and for the case of Riemann-Liouville we use the following definition:

$$\mathbb{D}_x^\alpha(f(x)) = \frac{1}{\Gamma(n-1)}\int_0^x (x-t)^{n-\alpha-1}\frac{d^nf(t)}{dt^n}dt, \qquad (2)$$

We proceed with the rest as follows:

Definition 2.1. Let $f\in L_1(a,b)$, then the Cauchy formulae is given by

$$\int_a^x\int_a^{x_{n-1}}...\int_a^{x_1} f(t)dtdx_1...dx_{(n-1)} = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}f(t)dt \qquad (3)$$

for $n \in$ N and $\alpha \in$ R$^+$.

The Riemann-Liouville fractional integrals are defined below.

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, x > a \quad (4)$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, x < b \quad (5)$$

From 3, let $n = \alpha$, for $\alpha \in$ R$^+$, and $(n - 1)! = \Gamma(\alpha)$ proofs (4) and (5). Oldham and Spanier. (1974) The following are some properties of the Riemann-Liouville fractional integrals.

Definition 2.2. Let $\alpha,\beta \in$ R and $f \in (a,b)$ then

$$I^{\beta}_{a+}I^{\alpha}_{a+}f(x) = I^{\alpha}_{a+}I^{\beta}_{a+}f(x) = I^{\alpha+\beta}_{a+}\ f(x) \quad (6)$$

This property is usually called semi-group property.

Definition 2.3. Let $\alpha \in$ R$^+$ and $f(x)$ is integrable, then

$$\mathbb{D}^{\alpha}_q I^{\alpha}_q f(x) = f(x) \quad (7)$$

Lemma 2.1. *Annaby and Mansour (2012) Let $f \in (0,a]$ and $\alpha \in$ R$^+$ for all $x \in (0,\infty]$. Then*

$$\lim_{\alpha\to 0+} I^{\alpha}_q f(x) = f(x). \quad (8)$$

In the theory of q-calculus, Ernst (2012) for a real parametera $q \in$ R$^+$\1, a q-real number$[a]_q$ is defined by $[a]_q = \frac{1-q}{1-q}, (a \in \mathbb{R})$.

Suppose $(a - b)^k$ is a power funtion, then the q-analog of this power function is given by

k−1

$$(a - b)^{(k)} = \Upsilon(a - bq^i) \quad (9)$$

i=0

where ($k \in$ N;$a,b \in$ R)

The natural expansion to reals are define by $(a-b)^{(\alpha)} = a^{\alpha}\frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}$, and $(a;q)_\alpha = \frac{(a;q)-\infty}{(aq^\alpha;q)_\infty}$.

From Jackson (1910) we state the following equations. The q-diferential defined as

$$d_q f(x) = f(qx) - f(x), \quad (10)$$

and

The q- derivative is defined as

$$D_q f(x := \frac{d_q f(qx)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}. \quad (11)$$

D_q is a linear operator. Thus

$$D_q(af(x) + bg(x)) = \frac{af(qx) + bg(qx) - af(x) - bg(x)}{(q-1)x}, \quad (12)$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}, \quad (13)$$

$$\frac{d_q}{d_q x}(f(x)g(x)) = f(x)\frac{d_q g}{dx} + g(x)\frac{d_q f}{dx}. \quad (14)$$

The Function $F(x)$ is q-antiderivative of $f(x)$ provided $D_qF(x) = f(x)$ Thus the q-antiderivative is defined as

$$F(x) = \int f(x)d_qx. \tag{15}$$

For definate integral, it is defined as

$$\int_a^b f(x)d_qx = \int_o^b f(x)d_qx - \int_o^a f(x)d_qx, \tag{16}$$

where $[a,b]$ are the limits of the integral. Similarlly, the q- analog of the integration by parts is defined as

$$\int_a^b f(x)(D_qg)(x)d_qx = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_qf)f(x)d_qx \tag{17}$$

Also see Oney (2007) Hasan et al. (2019),Iddrisu (2018),Nantomah (2017) Nantomah et al.

(2018),Freihet et al. (2019); **?**Ajega-Akem et al. (2019).

Definition 2.4. Let $\alpha \in$ C the Euler Gamma function define by

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt \tag{18}$$

Definition 2.5. Let $z,w,\in$ C for all $Re(z) > 0$ and $Re(w) > 0$, then

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt \tag{19}$$

and
$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$

Definition 2.6. Let $f(x,y,z)$ be a contineous and integrable funtion with respect y and z, then the change of order of integration is given by

$$\int_a^x \int_a^y f(x,y,z)dzdy = \int_a^x \int_z^x f(x,y,z)dydz \tag{20}$$

III. RESULTS AND DISCUSSION

We start this section with the following lemma.

Lemma 3.1. *Let $f \in L_1(a,b)$, then*

$$f(\xi) = \frac{1}{2\Gamma(\alpha)} \lim_{\alpha\to 0+} \int_a^b (t-\xi)^{\alpha-1}f(t)dt \tag{21}$$

for all $t \neq \xi$ and $(a,b) \in$ R.

Proof. Adding (4) and (5) yields

$$I^\alpha_{a+}f(\xi) + I^\alpha_{b-}f(\xi) = \frac{1}{\Gamma(\alpha)}\int_a^\xi (\xi-t)^{\alpha-1}f(t)dt + \frac{1}{\Gamma(\alpha)}\int_\xi^b (t-\xi)^{\alpha-1}f(t)dt, \tag{22}$$

for $\xi < b, \xi > a$, $R(\alpha) > 0$.

Taking limit as $\alpha \to 0+$, yields

$$\lim_{\alpha\to 0+}(I^{\alpha}_{a+}f(\xi)+I^{\alpha}_{b-}f(\xi))=\lim_{\alpha\to 0+}\left(\frac{1}{\Gamma_q(\alpha)}\int_a^{\xi}(\xi-tq)^{\alpha-1}f(t)d_qt+\frac{1}{\Gamma(\alpha)}\int_{\xi}^{b}(t-\xi)^{\alpha-1}f(t)dt\right) \quad (23)$$

$$\lim_{\alpha\to 0+}I^{\alpha}_{a+}f(\xi)+\lim_{\alpha\to 0+}I^{\alpha}_{b-}f(\xi)=$$
$$\lim_{\alpha\to 0+}\left(\frac{1}{\Gamma(\alpha)}\int_a^{\xi}(\xi-t)^{\alpha-1}f(t)dt+\frac{1}{\Gamma(\alpha)}\int_{\xi}^{b}(t-\xi)^{\alpha-1}f(t)dt\right) \quad (24)$$

Applying (8) we have

$$f(\xi)+f(\xi)=\lim_{\alpha\to 0+}\left(\frac{1}{\Gamma(\alpha)}\int_a^{\xi}(\xi-t)^{\alpha-1}f(t)dt+\frac{1}{\Gamma_q(\alpha)}\int_{\xi}^{b}(t-\xi)^{\alpha-1}f(t)dt\right) \quad (25)$$

Let $\alpha = 2n-1$, for $n = 1,2,...,$ then $(\xi - t)^{\alpha-1} = (t-\xi)^{\alpha-1}$.

Implies

$$2f(\xi)=\frac{1}{\Gamma(\alpha)}\lim_{\alpha\to 0+}\left(\int_a^{\xi}(t-\xi)^{\alpha-1}f(t)dt+\int_{\xi}^{b}(t-\xi)^{\alpha-1}f(t)dt\right) \quad (26)$$

(27)

Hence

$$2f(\xi)=\frac{1}{\Gamma(\alpha)}\lim_{\alpha\to 0+}\left(\int_a^{b}(t-\xi)^{\alpha-1}f(t)dt\right)$$

□

Corollary 3.2. *Let f be a continuous function on (a,b), then*

$$f(\xi)=\frac{1}{2}\lim_{\alpha\to 0+}\int_a^b f(t)dt. \quad (29)$$

for all tq 6= ξ and $(a,b) \in$ R.

Proof. From (28), put $\alpha = 1$, then the corollary is proved as required. □

Lemma 3.3. *Let* $f \in L_1(a,b)$, *then*

$$f(\xi)\le\frac{1}{2\Gamma(\alpha)}\lim_{\alpha\to 0+}\int_a^b|(t-\xi)|^{\alpha-1}f(t)dt \quad (30)$$

for all t 6= ξ and $(a,b) \in$ R.

Proof. From (25) let $\alpha = 2n$, for $n = 1,2,...$

then $(t-\xi)^{\alpha-1}$ 6= $(\xi-t)^{\alpha-1}$ but $|(t-\xi)| = |(\xi-t)|$ as required. □

Theorem 3.4. *Let f be a continuous function on (a,b), then*

$$f(\xi)=\frac{1}{2\Gamma_q(\alpha)}\lim_{\alpha\to 0+}D_q((t-\xi)^{(\alpha-1)})\left(\frac{[(t-\xi)^{\alpha-1}f(t)]_a^b}{D_q((t-\xi)^{(\alpha-1)})}-(I_{q,\xi}f)(t)+(I_{q,a}f)(t)\right). \quad (31)$$

Proof. Applying q-integration by parts to (28) we have

$$f(\xi)=\frac{1}{2\Gamma_q(\alpha)}\lim_{\alpha\to 0+}\left([(t-\xi)^{\alpha-1}f(t)]_a^b-[\alpha-1]_q(tq-\xi)^{\alpha-2}\times|(1-q)t\sum_{(j=0)}^{\infty}q^jf(q^jt)|_a^b\right) \quad (32)$$

$$f(\xi)=\frac{1}{2\Gamma_q(\alpha)}\lim_{\alpha\to 0+}\left([(t-\xi)^{\alpha-1}f(t)]_a^b\right)-$$

$$\frac{1}{2\Gamma_q(\alpha)}\lim_{\alpha\to 0+}\left([\alpha-1]_q(tq-\xi)^{\alpha-2}(1-q)b\sum_{j=0}^{\infty}q^jf(q^jb)\right) \quad (33)$$
$$+\frac{1}{2\Gamma_q(\alpha)}\lim_{\alpha\to 0+}\left([\alpha-1]_q(tq-\xi)^{\alpha-2}(1-q)a\sum_{j=0}^{\infty}q^jf(q^ja)\right) \quad (34)$$

$$f(\xi)=\frac{1}{2\Gamma_q(\alpha)}\lim_{\alpha\to 0+}\left(((t-\xi)^{\alpha-1}f(t)]_a^b)-[\alpha-1]_q(tq-\xi)^{\alpha-2}(1-q)b\sum_{j=0}^{\infty}q^jf(q^jb)\right)+$$
$$\frac{1}{2\Gamma_q(\alpha)}\lim_{\alpha\to 0+}\left([\alpha-1]_q(tq-\xi)^{\alpha-2}(1-q)a\sum_{j=0}^{\infty}q^jf(q^ja)\right).$$

But $(I_{q,a}f)(t) = \int_a^{\xi} f(t)d_qt$, $(I_{q,\xi}f)(t) = \int_{\xi}^{b} f(t)d_qt$ and

$D_q((t-\xi)^{(\alpha-1)}) = [\alpha-1]_q(tq-\xi)^{\alpha-2}$

$$f(\xi) = \frac{1}{2\Gamma_q(\alpha)} \lim_{\alpha\to 0+} \left(((t-\xi)^{\alpha-1}f(t)]_a^b) - D_q((t-\xi)^{(\alpha-1)})(I_{q,\xi}f)(t) \right) + \frac{1}{2\Gamma_q(\alpha)} \lim_{\alpha\to 0+} \left(D_q((t-\xi)^{(\alpha-1)})(I_{q,a}f)(t) \right). \quad (35)$$

This simplifies to

$$f(\xi) = \frac{1}{2\Gamma_q(\alpha)} \lim_{\alpha\to 0+} D_q((t-\xi)^{(\alpha-1)}) \left(\frac{[(t-\xi)^{\alpha-1}f(t)]_a^b}{D_q((t-\xi)^{(\alpha-1)})} - (I_{q,\xi}f)(t) + (I_{q,a}f)(t) \right) \quad (36)$$

as required. □

Theorem 3.5. *Let f be a continuous function on (a,b), then for all α = 2n,*

$$f(\xi) \leq \frac{1}{2\Gamma_q(\alpha)} \lim_{\alpha\to 0+} D_q|((t-\xi)|^{(\alpha-1)}) \left(\frac{[|(t-\xi)|^{\alpha-1}f(t)]_a^b}{D_q(|(t-\xi)|^{(\alpha-1)})} - (I_{q,\xi}f)(t) + (I_{q,a}f)(t) \right) \quad (37)$$

for all t 6= ξ and $(a,b) \in$ R.

Proof. Applying q-integration by parts to (30) gives

$$f(\xi) \leq \frac{1}{2\Gamma_q(\alpha)} \lim_{\alpha\to 0+} \left((|(t-\xi)|^{\alpha-1}f(t)]_a^b) - D_q(|(t-\xi)|^{\alpha-1})(I_{q,\xi}f)(t) \right) + \frac{1}{2\Gamma_q(\alpha)} \lim_{\alpha\to 0+} \left(D_q(|(t-\xi)|^{\alpha-1})(I_{q,a}f)(t) \right). \quad (38)$$

Simplifying this prove the theorem.

IV. CONCLUSIONS

In this write up, some new inequalities involving Riemann-Liouville fractional integral inequalities using q-Calculus, were presented. The research results established was realised through a property usually called semi-group property coupled with applying q-integration by parts and simplifying yielded the desired results.

➢ *Conflict of Interests*

The authors declare that they have no conflict of interests.

➢ *Authors Contribution*

Both authors contributed inthe write up. The final authors read and approved the final draft Acknowledgments

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