ISSN No:-2456-2165

An Application of Spacer Matrix based Matrix chain propagation: Associating Complex Infinite Sequences with Elements of M(r,c) Subsets of Complex Matrix Spaces of Order m by n, Where m≠n

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Abstract:- The research article presents a mathematical formalism to associate Infinite sequences of complex numbers with matrices belonging to $M(r,c)$ subsets of strictly rectangular complex matrix spaces. This is achieved by using the Spacer matrix (as defined in [2]) based Matrix chain propagation. The formalism is discussed and clarified with appropriate numerical examples

Keywords:- *Global Mass Factor of a Matrix, Effective Global Mass Factor of a Matrix, Global Alignment Factor of a Matrix, Spacer Matrix, Generalized Matrix Multiplication.*

Notations

- $M_{m\times n}(C)$ denotes the Complex Matrix space of Matrices of order m by n
- $R(A)$ denotes the Global Mass Factor associated with the matrix $A_{m\times n}$
- $R_0(A)$ denotes the Effective Global Mass Factor associated with the matrix $A_{m\times n}$
- $C(A)$ denotes the Global Alignment Factor associated with the matrix $A_{m\times n}$
- $|c|$ denotes the modulus of the complex number c
- $c^{\bullet}$ denotes the complex conjugate of the complex number c
- $\{|e_1\rangle, |e_2\rangle, ..., |e_m\rangle\}$ denotes the standard Orthonormal basis in C^m and $\{|f_1\rangle, |f_2\rangle, ..., |f_n\rangle\}$ denotes the standard Orthonormal basis in C^n
- X^H denotes the Hermitian conjugate of the matrix X
- $M(r,c)$ Is a subset of the Complex Matrix space $M_{m\times n}(C)$ characterized by the numerical values of the Global Mass factor and Global alignment factor, r and c, respectively

$$|W\rangle = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_p \end{bmatrix}_{p\times 1}, \langle V| = [v_1^{\bullet} \quad v_2^{\bullet} \quad . \quad . \quad v_q^{\bullet}]_{1\times q}, B = [b_{ij}]_{q\times p}, \langle V|B|W\rangle = \sum_{i=1}^{q}\sum_{j=1}^{p} b_{ij} v_i^{\bullet} w_j$$

- $|m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{m\times 1}, \langle m| = [1 \quad 1 \quad . \quad . \quad 1]_{1\times m}$
- N denotes the set of all Natural numbers
- u denotes the Embedding dimension
- $X_{n\times m}$ denotes the Spacer Matrix associated with the complex matrix space $M_{m\times n}(C)$
- $P_{n\times u}, Q_{u\times m}$ are the component Matrices associated with the Spacer Matrix $X_{n\times m}$
- $I_{s\times s}$ denotes the Identity Matrix of order 's'
- $\max(a,b)$ denotes the maximum of the two inputs a and b, $a,b \in N$
- $|a-b|$ denotes the absolute value of the difference of the two inputs a and b, $a,b \in N$
- $B_{s\times s}{}^{t} = B\times B\times B\times .....\times B$ (t times), ' × ' denotes ordinary matrix multiplication
- C^{INF} denotes the set of all Infinite sequences of complex numbers

I. INTRODUCTION

The Spacer Matrix based Matrix Multiplication scheme [2], can be used to link up infinite (countable infinite) sequence of matrices belonging to a strictly rectangular (m≠n) complex matrix space. In this research article the focus is on the above being applied to M(r,c) subsets of the complex Matrix spaces $M_{m\times n}(C)$ ([1],[3]), i.e. subsets of complex matrices characterized by a given numerical value of the Global Mass and the Global Alignment factor. The chain propagation is determined by the spacer matrix elements, weightage terms arising out of modulus distribution and the composite phase terms associated with each degrees of freedom of the Matrix $A_{m\times n}\in M(r,c)$. Every Iteration in Chain propagation (Increment in Chain size) is captured through an inner product formation, i.e. as a complex scalar. Therefore, the Matrix $A_{m\times n}$ is associated with an infinite sequence of such complex scalars, such sequences themselves are elements of the set of all complex, infinite sequences.

The article presents the Mathematical formalism and illustrates the same through its application on the numerical examples discussed in [1].

II. MATHEMATICAL FRAMEWORK AND ASSOCIATED ANALYSIS

The following results, stated in [1], [2] and [3] are used to provide the groundwork for the formalism described in this research article:

- $A\in M_{m\times n}(C)$, $A=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\left|e_i\right\rangle\left\langle f_j\right|$, $a_{ij}=r_{ij}c_{ij}$, $r_{ij}=\left|a_{ij}\right|$, $c_{ij}\in C$, $\left|c_{ij}\right|=1$, we consider the following convention that in the case of zero matrix elements of matrix A: $a_{ij}=0\Rightarrow r_{ij}=0, c_{ij}=1$
- $R(A)=\sum_{i=1}^{m}\sum_{j=1}^{n}r_{ij}$, $R_0(A)=\sum_{i=1}^{m}\sum_{j=1}^{n}r_{ij}(1-\exp(-r_{ij}))$
- $C(A)=c_{11}c_{12}....c_{1n}c_{21}c_{22}....c_{2n}.........c_{m1}c_{m2}....c_{mn}=\prod_{i=1}^{m}\prod_{j=1}^{n}c_{ij}$

$C(A)\in C,\left|C(A)\right|=1,\forall A\in M_{m\times n}(C)$

- $M(r,c)\subset M_{m\times n}(C), M(r,c)=\left\{A\in M_{m\times n}(C)\,|\,A\neq 0_{m\times n}, R(A)=r, C(A)=c\right\}$ where we have the condition: $r>0, c\in C, \left|c\right|=1$
- $\lambda_{ij}=(\frac{r_{ij}}{r_0})(1-\exp(-r_{ij}))$, where $r_0=\sum_{i=1}^{m}\sum_{j=1}^{n}r_{ij}(1-\exp(-r_{ij}))$ is the numerical realization of the Effective Global Mass factor $R_0(A)$
- $\mu_{ij}=(\frac{r_{ij}}{r})$, where $r=\sum_{i=1}^{m}\sum_{j=1}^{n}r_{ij}$ is the numerical realization of the Global Mass factor $R(A)$
- $\lambda_{ij}\geq 0,\forall i=1,2,..m; j=1,2,....,n$, and we have : $\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}=1$
- $\mu_{ij}\geq 0,\forall i=1,2,..m; j=1,2,....,n$, and we have : $\sum_{i=1}^{m}\sum_{j=1}^{n}\mu_{ij}=1$
- $\alpha_i=c_{i1}c_{i2}.....c_{in}$, $\beta_j=c_{1j}c_{2j}.....c_{mj}$, Therefore $\alpha_i\beta_j=c_{i1}c_{i2}.....c_{in}c_{1j}c_{2j}.....c_{mj}$, $i=1,2,..m; j=1,2,....,n$, we have: $\alpha_i\beta_j\in C,\left|\alpha_i\beta_j\right|=1,\forall i=1,2,....m; j=1,2,....,n$

- $m,n \in N, m \neq n$, we have: $u = \max(m,n) + |m-n|, u > m, u > n$

$X_{n\times m} = P_{n\times u} Q_{u\times m}$, where:

$$P = \begin{bmatrix} I_{n\times n} & (\frac{1}{n})|n\rangle\langle u-n| \end{bmatrix}_{n\times u} = \left[\begin{array}{ccccc|ccccc} 1 & 0 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ 0 & 1 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 1/n & 1/n & . & . & 1/n \end{array}\right],$$

$$Q = \begin{bmatrix} I_{m\times m} \\ (\frac{1}{m})|u-m\rangle\langle m| \end{bmatrix}_{u\times m} = \left[\begin{array}{ccccc} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \\ \hline 1/m & 1/m & . & . & 1/m \\ 1/m & 1/m & . & . & 1/m \\ . & . & . & . & . \\ . & . & . & . & . \\ 1/m & 1/m & . & . & 1/m \end{array}\right]$$

We define the following:

$$\Lambda_{m\times n} = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij} |e_i\rangle\langle f_j| \ , \ \Omega_{m\times n} = \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_{ij} |e_i\rangle\langle f_j| \ , \ W(\alpha,\beta) = \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_i\beta_j |e_i\rangle\langle f_j| = |\alpha\rangle\langle\beta^{\bullet}|$$

Where: $|\alpha\rangle = \sum_{i=1}^{m} \alpha_i |e_i\rangle$, $\langle\beta^{\bullet}| = \sum_{j=1}^{n} \beta_j \langle f_j|$

$$(\Lambda X \Lambda X)_{m\times m} = \Lambda_{m\times n} P_{n\times u} Q_{u\times m} \Lambda_{m\times n} P_{n\times u} Q_{u\times m} \ , \ (X \Lambda X \Lambda)_{n\times n} = P_{n\times u} Q_{u\times m} \Lambda_{m\times n} P_{n\times u} Q_{u\times m} \Lambda_{m\times n}$$

$$(\Lambda X \Omega X)_{m\times m} = \Lambda_{m\times n} P_{n\times u} Q_{u\times m} \Omega_{m\times n} P_{n\times u} Q_{u\times m} \ , \ (X \Lambda X \Omega)_{n\times n} = P_{n\times u} Q_{u\times m} \Lambda_{m\times n} P_{n\times u} Q_{u\times m} \Omega_{m\times n}$$

$$(\Omega X \Lambda X)_{m\times m} = \Omega_{m\times n} P_{n\times u} Q_{u\times m} \Lambda_{m\times n} P_{n\times u} Q_{u\times m} \ , \ (X \Omega X \Lambda)_{n\times n} = P_{n\times u} Q_{u\times m} \Omega_{m\times n} P_{n\times u} Q_{u\times m} \Lambda_{m\times n}$$

$$(\Omega X \Omega X)_{m\times m} = \Omega_{m\times n} P_{n\times u} Q_{u\times m} \Omega_{m\times n} P_{n\times u} Q_{u\times m} \ , \ (X \Omega X \Omega)_{n\times n} = P_{n\times u} Q_{u\times m} \Omega_{m\times n} P_{n\times u} Q_{u\times m} \Omega_{m\times n}$$

We define: $W(\alpha,\beta|\Lambda,\Lambda|t) = (\Lambda X \Lambda X)^t W(\alpha,\beta)(X\Lambda X\Lambda)^t$, $W(\alpha,\beta|\Omega,\Omega|t) = (\Omega X \Omega X)^t W(\alpha,\beta)(X\Omega X\Omega)^t$, $W(\alpha,\beta|\Lambda,\Omega|t) = (\Lambda X \Omega X)^t W(\alpha,\beta)(X\Lambda X\Omega)^t$,

$$W(\alpha,\beta|\Omega,\Lambda|t) = (\Omega X \Lambda X)^t W(\alpha,\beta)(X\Omega X\Lambda)^t \ , \ t = 1,2,3,\ldots\ldots \ , \ t \in N$$

$$W(\alpha,\beta|t) = (\frac{1}{4})[W(\alpha,\beta|\Lambda,\Lambda|t) + W(\alpha,\beta|\Omega,\Omega|t) + W(\alpha,\beta|\Lambda,\Omega|t) + W(\alpha,\beta|\Omega,\Lambda|t)]$$

$t = 1,2,3,\ldots\ldots, t \in N$

We define: $\delta(0) = (\frac{1}{m.n})\langle\alpha|W(\alpha,\beta)|\beta^{\bullet}\rangle = (\frac{1}{m.n})\langle\alpha|\alpha\rangle\langle\beta^{\bullet}|\beta^{\bullet}\rangle = 1$

$$\delta(t) = (\frac{1}{m.n})\langle\alpha|W(\alpha,\beta\,|\,t)|\beta^{\bullet}\rangle \ , \ t = 1,2,3,......,t \in N$$

We have: $\bar{C}(A_{m\times n}) = \{\delta(1),\delta(2),.......,\delta(t),...........\}$, therefore, $\bar{C}(A_{m\times n}) \in C^{INF}$

The complete mapping process can be represented in terms of the transformation $\hat{\Pi}$:

$$\hat{\Pi}(\sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}c_{ij}\,|e_i\rangle\langle f_j|) = \{\delta(1),\delta(2),.......,\delta(t),...........\} \qquad \text{.... (eqn.)}$$

Numerical Examples

1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$, $A \in M(r=2, c=1)$, we have the following: $\Lambda = \Omega = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}_{2\times 3}$

$W(\alpha,\beta) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$, $(\Lambda X \Lambda X) = (\Lambda X \Omega X) = (\Omega X \Lambda X) = (\Omega X \Omega X) = (\frac{1}{72})\begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}_{2\times 2}$

$$(X\Lambda X\Lambda) = (X\Lambda X\Omega) = (X\Omega X\Lambda) = (X\Omega X\Omega) = (\frac{1}{72})\begin{bmatrix} 25 & 7 & 0 \\ 7 & 25 & 0 \\ 16 & 16 & 0 \end{bmatrix}_{3\times 3} ,$$

$W(\alpha,\beta\,|\,t) = W(\alpha,\beta\,|\,\Lambda,\Lambda\,|\,t) = W(\alpha,\beta\,|\,\Omega,\Omega\,|\,t) = W(\alpha,\beta\,|\,\Lambda,\Omega\,|\,t) = W(\alpha,\beta\,|\,\Omega,\Lambda\,|\,t), \forall t = 1,2,3,......,t \in N$
$\bar{C}(A_{2\times 3}) = \{0.197531, 0.039018, 0.007707, 0.001522, 0.000301,........\}$ .... (Each term in the sequence approximated to 6 places of decimal)

2) $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$, $B \in M(r=2, c=1)$, we have the following: $\Lambda = \Omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

3)

$W(\alpha,\beta) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$, $(\Lambda X \Lambda X) = (\Lambda X \Omega X) = (\Omega X \Lambda X) = (\Omega X \Omega X) = (\frac{1}{36})\begin{bmatrix} 49 & 7 \\ 0 & 0 \end{bmatrix}_{2\times 2}$,

$$(X\Lambda X\Lambda) = (X\Lambda X\Omega) = (X\Omega X\Lambda) = (X\Omega X\Omega) = (\frac{1}{36})\begin{bmatrix} 49 & 0 & 0 \\ 7 & 0 & 0 \\ 28 & 0 & 0 \end{bmatrix}_{3\times 3} ,$$

$W(\alpha,\beta\,|\,t) = W(\alpha,\beta\,|\,\Lambda,\Lambda\,|\,t) = W(\alpha,\beta\,|\,\Omega,\Omega\,|\,t) = W(\alpha,\beta\,|\,\Lambda,\Omega\,|\,t) = W(\alpha,\beta\,|\,\Omega,\Lambda\,|\,t), \forall t = 1,2,3,......,t \in N$

ISSN No:-2456-2165

$\bar{C}(B_{2\times 3}) = \{0.604938, 1.120723, 2.076277, 3.846560, 7.126228, ...\}$ .... (Each term in the sequence approximated to 6 places of decimal)

4) $C = \begin{bmatrix} -i & 0 & 0 \\ 0 & +i & 0 \end{bmatrix}_{2\times 3}$, $C \in M(r=2, c=1)$, we have the following: $\Lambda = \Omega = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}_{2\times 3}$,

5)

$$W(\alpha,\beta) = \begin{bmatrix} -1 & 1 & -i \\ 1 & -1 & +i \end{bmatrix}_{2\times 3}, \ (\Lambda X \Lambda X) = (\Lambda X \Omega X) = (\Omega X \Lambda X) = (\Omega X \Omega X) = (\frac{1}{72}) \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}_{2\times 2},$$

$$(X \Lambda X \Lambda) = (X \Lambda X \Omega) = (X \Omega X \Lambda) = (X \Omega X \Omega) = (\frac{1}{72}) \begin{bmatrix} 25 & 7 & 0 \\ 7 & 25 & 0 \\ 16 & 16 & 0 \end{bmatrix}_{3\times 3},$$

$$W(\alpha,\beta \mid t) = W(\alpha,\beta \mid \Lambda,\Lambda \mid t) = W(\alpha,\beta \mid \Omega,\Omega \mid t) = W(\alpha,\beta \mid \Lambda,\Omega \mid t) = W(\alpha,\beta \mid \Omega,\Lambda \mid t), \forall t = 1,2,3,......, t \in N$$

$\bar{C}(C_{2\times 3}) = \{0.041667, 0.002604, 0.000163, 0.000010, ...\}$ .... (Each term in the sequence approximated to 6 places of decimal)

6) $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$, $D \in M(r=6, c=1)$, we have the following: $\Lambda = \Omega = (\frac{1}{6}) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$,

$$W(\alpha,\beta) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}, \ (\Lambda X \Lambda X) = (\Lambda X \Omega X) = (\Omega X \Lambda X) = (\Omega X \Omega X) = (\frac{2}{9}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2},$$

$$(X \Lambda X \Lambda) = (X \Lambda X \Omega) = (X \Omega X \Lambda) = (X \Omega X \Omega) = (\frac{4}{27}) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3},$$

$$W(\alpha,\beta \mid t) = W(\alpha,\beta \mid \Lambda,\Lambda \mid t) = W(\alpha,\beta \mid \Omega,\Omega \mid t) = W(\alpha,\beta \mid \Lambda,\Omega \mid t) = W(\alpha,\beta \mid \Omega,\Lambda \mid t), \forall t = 1,2,3,......, t \in N$$

$\bar{C}(D_{2\times 3}) = \{0.197531, 0.039018, 0.007707, 0.001522, 0.000301, ...\}$ .... (Each term in the sequence approximated to 6 places of decimal)

7) $E = \begin{bmatrix} 1 & +i & -i \\ -i & 1 & +i \end{bmatrix}_{2\times 3}$, $E \in M(r=6, c=1)$, we have the following: $\Lambda = \Omega = (\frac{1}{6}) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$,

$$W(\alpha,\beta) = \begin{bmatrix} -i & +i & 1 \\ -i & +i & 1 \end{bmatrix}_{2\times 3}, \ (\Lambda X \Lambda X) = (\Lambda X \Omega X) = (\Omega X \Lambda X) = (\Omega X \Omega X) = (\frac{2}{9}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2},$$

ISSN No:-2456-2165

$$(X\Lambda X\Lambda) = (X\Lambda X\Omega) = (X\Omega X\Lambda) = (X\Omega X\Omega) = (\frac{4}{27})\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3},$$

$$W(\alpha,\beta\,|\,t) = W(\alpha,\beta\,|\,\Lambda,\Lambda\,|\,t) = W(\alpha,\beta\,|\,\Omega,\Omega\,|\,t) = W(\alpha,\beta\,|\,\Lambda,\Omega\,|\,t) = W(\alpha,\beta\,|\,\Omega,\Lambda\,|\,t), \forall t = 1,2,3,......, t \in N$$

$\bar{C}(E_{2\times 3}) = \{0.021948, 0.004335, 0.000856, 0.000169, 0.000033, ...\}$ .... (Each term in the sequence approximated to 6 places of decimal)

III. DISCUSSION AND CONCLUSION

The article is an attempt to tie together the concepts of Spacer Matrix based Matrix Chain propagation with matrices belonging to $M(r,c)$ subsets. Using an Initiator matrix $W(\alpha,\beta)$ and Chain propagation matrix pairs $(\Lambda X\Lambda X, X\Lambda X\Lambda)$, $(\Lambda X\Omega X, X\Lambda X\Omega)$, $(\Omega X\Lambda X, X\Omega X\Lambda)$, $(\Omega X\Omega X, X\Omega X\Omega)$, possibly infinite sequences of matrices belonging to $M_{m\times n}(C)$ are generated.

Under the effect of inner product formation, this results in infinite sequence of complex numbers.

In examples 1 through 3, the matrices belong to the $M(r=2, c=1)$ subset, $A_{2\times 3}$ and $B_{2\times 3}$ are associated with the same Initiator but different Propagator matrices, $A_{2\times 3}$ and $C_{2\times 3}$ are associated with the same Propagator but different Initiator matrices. It can be observed that they are associated with different sequences, with $\bar{C}(A_{2\times 3})$ and $\bar{C}(C_{2\times 3})$ appearing to converge absolutely to zero in the limit while the sequence $\bar{C}(B_{2\times 3})$ appears to be divergent in absolute sense.

In examples 4 and 5, the matrices belong to the $M(r=6, c=1)$ subset; they differ from each other only in terms of the Initiator Matrix, It can be observed that both of them appear to converge to zero in absolute sense. It is evident from an examination of the presented formalism that the limiting behavior of these sequences is dictated by the overlap behavior of the participating matrices and vectors in the sequence, this aspect will be studied and analyzed in more details in subsequent follow up studies, and also focus will be made on applicability of the presented mathematical formalism in solving real world problems.

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