ISSN No:-2456-2165

Kharrat-Toma Transform and its Application in Solving Some Ordinary Differential Equations with Initial Boundary Conditions

MUSTAPHA R.A.[1] , SALAU A.M.[2] , BABATUNDE Idowu.[3] OGABI C.O.[4] , IDOWU G.A.[5]
[1,2,5] Department of Mathematics, Lagos State University, Ojo, Lagos, Nigeria
[3,4] Department of Physics, Lagos State University, Ojo, Lagos, Nigeria

Abstract:- In this paper, we adopted a new integral transform called Kharrat-Toma Transform which can be considered to be a basis for a number of potential new integral transforms. Some fundamental properties about this new integral transform were used in this work, includes the existence theorem, transportation theorem, convolution theorem and inversion equation. The major advantage of this new technique is that it solves ordinary differential equation with variable and constant coefficients. Some relevant examples were solved to show the efficiency of this technique.

***Keywords**:- Kharrat-Toma Transform, Ordinary Differential Equations, Exact Solution.*

I. INTRODUCTION

Differential equations plays vital role in engineering, physics, mathematics, Applied mathematics, chemistry and physiology Applications; In the research work, we suggest to develop a new technique for obtaining solution which approximate the exact solution.

There are many integral transforms widely used to solve the differential equation and thus there are several works on the differential transform such as laplace transform was introduced by P.S. Laplace in 1780's [1], laplace transform is the oldest integral transform and the most widely used. The Stieltjes in [2], was the first to give a systematic formulation of the mellin transformation in [3], The Mohomd Transform was introduced by mohomd M.Mahgoub in [4], S. Ahmad et al proposed a new integral transform to solve higher order linear Laguerre and Hermite differential equations in [5], D. Hilbert Suggested the Hilbert transform in [7], J. Radon founded the radon transform in [8], Laguerre transform by Edmond Laguerre in [9], G.K. watugula introduced the Sumudu transform in [10]. Natural Transform was initiated by Khan and Khan in [11]. The Aboodh transform was presented by Khalid .S. Aboodh in [12]. The Elzaki transform was presented by Tarig M. Elzaki in [13]. The new integral transform "M-transform" was suggested by Srivastava in [14]. The ZZ transform was devised by Zafar in [15], A. Kamal and H. Sedeeq proposed the Kamal Transform in [16], the Yang transform was introduced by Xiao-jun Yang in [17]. And finally R. Saaduh et al introduced ARA transform in [18].

Kharrat et al also interested in integral transform methods, where they applied the Differential transform to solve boundary value problems represented by differential equations from higher orders and also to solve a system of differential equation [19-21]. In addition, they suggested hybridization the homotopy perturbation method with sumundu transform to solve initial value problems for nonlinear partial differential equation [22], They also introduced the hybridization of the Natural transform method with the homotopy perturbation method to solve Van Der pol oscillator problem [23].

The purpose of this work is to show the efficiency and applicability of the new integral transform and applied it to solve Ordinary differential equations with variable and constant coefficients as proposed by Kharrat et al in [24]. The rest of the paper is as follows: we present the basic idea of Kharrat-Toma transform in Section 2. In Section 3, Kharrat-Toma transform of some functions is introduced and we proof some properties, In section 4, the application for solving ordinary differential equation is shown and conclusion in section 5.

Kharrat-Toma Transform: Definition 1. The function $f(x)$ is said to have exponential order on every finite interval in $[0,+\infty)$ If there exist a positive number *M* that satisfying: $|f(x)| \le Me^{\propto x}, \ M > 0, \propto > 0, \quad \forall x \ge 0$

Definition 2: The Kharrat-Toma integral transform and inversion is defined by.

$$\boldsymbol{B[f(x)] = G(s) = s^3 \int_0^\infty f(x)\, e^{\frac{-x}{s^2}}\, dx \ , x \ge 0}$$

$$\boldsymbol{f(x) = B^{-1}[G(S)] = B^{-1}\left[s^3 \int_0^\infty f(x)\, e^{\frac{-x}{s^2}}\, dx\right]}$$

The B integral transform states that, if $f(x)$ is piecewise continuous on $[0,+\infty)$ and has exponential order. The $\boldsymbol{B^{-1}}$ will be the inverse of the B integral transform.

Theorem 1: [Sufficient Condition for Existence of a Kharrat-Toma Transform]: The Kharrat-Toma transform $\boldsymbol{B[f(x)]}$ exists if it has exponential order and $\int_0^b |\boldsymbol{f(x)}|\,\boldsymbol{dx}$ exists for any $b > 0$.
Proof:

$$s^3\int_0^\infty \left|f(x)\,e^{\frac{-x}{s^2}}\right|dx = s^3\int_0^n \left|f(x)\,e^{\frac{-x}{s^2}}\right|dx + s^3\int_n^\infty \left|f(x)\,e^{\frac{-x}{s^2}}\right|dx$$

$$\leq s^3\int_0^n |f(x)|\,dx + s^3\int_0^\infty |f(x)|\,e^{\frac{-x}{s^2}}dx$$

$$\leq s^3\int_0^n |f(x)|\,dx + Ms^3\int_0^\infty e^{\propto x}\,e^{\frac{-x}{s^2}}dx$$

$$= s^3\int_0^n |f(x)|\,dx + Ms^3\int_0^\infty e^{-\left(\frac{1}{s^2}-\propto\right)x}\,dx$$

$$= s^3\int_0^n |f(x)|\,dx + \frac{Ms^3}{-\left(\frac{1}{s^2}-\propto\right)}\lim_{B\to\infty}\left[e^{-\left(\frac{1}{s^2}-\propto\right)x}\right]_0^B \;;\; \frac{1}{s^2} > \propto$$

$$= s^3\int_0^n |f(x)|\,dx + \frac{Ms^3}{\frac{1}{s^2}-\propto}$$

The first integral exists, and the second term is finite for $\frac{1}{s^2} > \propto$, so the integral $\boldsymbol{s^3\int_0^\infty f(x)\,e^{\frac{-x}{s^2}}dx}$ converges absolutely and the Kharrat-Toma $\boldsymbol{B[f(x)]}$ exists.

Kharrat-Toma Transform of Some Functions: In this section we find Kharrat-Toma transform of some functions;

$$f(x) = 1 \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = s^5 \qquad (1)$$

$$f(x) = x^n \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = s^{2n+5}.n! \qquad (2)$$

$$f(x) = sin(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \frac{ks^7}{1+k^2s^4} \qquad (3)$$

$$f(x) = cos(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \frac{s^5}{1+k^2s^4} \qquad (4)$$

$$f(x) = sinh(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \frac{ks^7}{1-k^2s^4} \qquad (5)$$

$$f(x) = cosh(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \frac{s^5}{1-k^2s^4} \qquad (6)$$

Proof:

$$B[1]=s^3\int_0^\infty e^{\frac{-x}{s^2}}\,dx = -s^5\lim_{B\to\infty}\left[e^{\frac{-x}{s^2}}\right]_0^B = s^5$$

Where $s^2>0$

(1)

$$B[x^n]=s^3\int_0^\infty x^n e^{\frac{-x}{s^2}}\,dx$$
$$u=x^n \Rightarrow du=nx^{n-1}dx$$
$$dv=e^{\frac{-x}{s^2}}\mathrm{d}x \Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

(2)

Then we get,

$$B[x^n]=s^3\left[-s^2x^n e^{\frac{-x}{s^2}}\Big|_0^\infty + ns^2\int_0^\infty x^{n-1}e^{\frac{-x}{s^2}}\,dx\right]$$
$$= ns^5\int_0^\infty x^{n-1}e^{\frac{-x}{s^2}}\,dx$$
$$u=x^{n-1}\Rightarrow du=(n-1)x^{n-2}dx$$
$$dv=e^{\frac{-x}{s^2}}\mathrm{d}x \Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Yields

$$B[x^n]=ns^5\left[-s^2x^{n-1}e^{\frac{-x}{s^2}}\Big|_0^\infty + (n-1)s^2\int_0^\infty x^{n-2}e^{\frac{-x}{s^2}}\,dx\right]$$
$$= n(n-1)s^7\int_0^\infty x^{n-2}e^{\frac{-x}{s^2}}\,dx=\cdots= s^{2n+5}.n!$$

Where $s^2>0$

(3)

$$B[sin(kx)]=s^3\int_0^\infty sin(kx)e^{\frac{-x}{s^2}}\,dx$$
$$u=sin(kx)\Rightarrow du=kcos(kx)dx$$
$$dv=e^{\frac{-x}{s^2}}\mathrm{d}x \Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Then we get,

$$B[sin(kx)]=s^3\left[-s^2sin(kx)\,e^{\frac{-x}{s^2}}\Big|_0^\infty + ks^2\int_0^\infty cos(kx)e^{\frac{-x}{s^2}}\,dx\right]$$
$$=ks^5\int_0^\infty cos(kx)e^{\frac{-x}{s^2}}\,dx\ ;\ s^2>0$$
$$u=cos(kx)\Rightarrow du=-ksin(kx)dx$$
$$dv=e^{\frac{-x}{s^2}}\mathrm{d}x \Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Yields

$$B[sin(kx)]=k\,s^5\left[-s^2cos(kx)\,e^{\frac{-x}{s^2}}\Big|_0^\infty - ks^2\int_0^\infty sin(kx)e^{\frac{-x}{s^2}}\,dx\right]$$
$$=ks^5\left[s^2-\frac{k}{s}B[sin(kx)]\right]$$

Then we get,

$$B[sin(kx)]=\frac{ks^7}{1+k^2s^4}$$

(4)

Proof in the same way as in (3)

$$B[sinh(kx)]=s^3\int_0^\infty sinh(kx)e^{\frac{-x}{s^2}}\,dx$$
$$u=sinh(kx)\Rightarrow du=kcosh(kx)dx$$
$$dv=e^{\frac{-x}{s^2}}\mathrm{d}x \Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Then we get,

$$B[sinh(kx)]=s^3\left[-s^2sinh(kx)\,e^{\frac{-x}{s^2}}\Big|_0^\infty + ks^2\int_0^\infty cosh(kx)e^{\frac{-x}{s^2}}\,dx\right]$$
$$=ks^5\int_0^\infty cosh(kx)e^{\frac{-x}{s^2}}\,dx\ ;\ s^2>0$$

(5)

$$u=cosh(kx)\Rightarrow du=ksinh(kx)dx$$
$$dv=e^{\frac{-x}{s^2}}\mathrm{d}x \Rightarrow v=-s^2e^{\frac{-x}{s^2}}$$

Yields

$$B[sinh(kx)]=k\,s^5\left[-s^2cosh(kx)\,e^{\frac{-x}{s^2}}\Big|_0^\infty + ks^2\int_0^\infty sinh(kx)e^{\frac{-x}{s^2}}\,dx\right]$$
$$=ks^5\left[s^2+\frac{k}{s}B[sinh(kx)]\right]$$

Then we get,

$$B[sinh(kx)]=\frac{ks^7}{1-k^2s^4}$$

(6) Proof the same way as in (5)

Theorem 2:

Let $B[f_1(x)]=G_1(s),\ldots,B[f_n(x)]=G_n(s)$ **and the constant** $c_1\ldots,c_n$, **then**

$$B\left[\sum_{i=1}^{n}c_i\,f_i(x)\right]=\sum_{i=1}^{n}c_i\,B\,[f_i(x)]$$

Proof:

$$B\left[\sum_{i=1}^{n}c_i\,f_i(x)\right]=s^3\int_0^\infty\sum_{i=1}^{n}c_i\,f_i(x)\,e^{\frac{-x}{s^2}}dx$$
$$=\sum_{i=1}^{n}c_i\left(s^3\int_0^\infty f_i(x)\,e^{\frac{-x}{s^2}}dx\right)$$
$$=\sum_{i=1}^{n}c_iB[f_i(x)]$$

Translation Property, First Shifting Property, Convolution Theorem, Kharrat-Toma Transform of $x^nf(x); n\geq 1$, Kharrat-Toma Transform of Derivatives was exhaustively proved in [24].

Application: In this section, we introduce the methodology of application of Kharrat-Toma transform for solving initial value problem. This new integral transform can be used as an effective tool for solving ordinary differential equations with initial conditions, we also compared the result of Kharrat-Toma transform with the traditional way of solving ordinary differential equations(Exact solution). We considered some examples to show the use and efficiency of this integral transform.

Example 1:

We consider the initial value problem

$$\begin{cases} y' = 2y + 3 \\ y(0) = 1 \end{cases}$$
(1)

Using analytical approach (Exact solution),
By using integrating factor(IF) $e^{\int f(x)dx}$
$y'(x) + f(x)y(x) = g(x)$
$e^{\int -2\,dx} = e^{-2x}$
$e^{-2x}y'(x) - 2e^{-2x}y(x) = 3e^{-2x}$
$\frac{d}{dx}e^{-2x}y = 3e^{-2x}$
$\int \frac{d}{dx}e^{-2x}y = \int 3e^{-2x}\,dx$
$e^{-2x}y(x) = -\frac{3}{2}e^{-2x} + c$
$y = 1$ When $x = 0$
$1(1) = -\frac{3}{2} + c$
$c = \frac{5}{2}$
$\therefore y(x) = \frac{5}{2}\,e^{2x} - \frac{3}{2}$

Using Kharrat-Toma for example 1,
$y' = 2y + 3$
$y' - 2y = 3$
$B[y' - 2y] = B[3]$
$B[y'] - 2B[y] = B[3]$

Applying the kharrat-Toma transform on (1), we get

$$\frac{1}{s^2}G(S) - s^3y(0) - 2G(s) = 3s^5$$
$$\frac{1}{s^2}G(S) - s^3 - 2G(s) = 3s^5$$
$$G(S)\left[\frac{1}{s^2} - 2\right] = 3s^5 + s^3$$
$$G(S) = \frac{3s^7 + s^5}{1 - 2s^2}$$
$$= \frac{3s^7}{1 - 2s^2} + \frac{s^5}{1 - 2s^2}$$
$$= \frac{s^5}{1} - \frac{s^5 - 5s^7}{1 - 2s^2} + \frac{s^5}{1 - 2s^2}$$
$$= -\frac{3}{2}s^5 + \frac{5}{2}\frac{s^5}{(1 - 2s^2)}$$
$$= B\left[-\frac{3}{2}\right] + B\left[\frac{5}{2}e^{2x}\right]$$

Applying the inverse Kharrat-Toma transform then the exact solution for the Initial value problem is

$$y(x) = -\frac{3}{2} + \frac{5}{2}\,e^{2x}$$

Example 2:
We consider the initial value problem
$$\begin{cases} y' = 6y + 1 \\ y(0) = 1 \end{cases}$$
(2)

Using analytical approach (Exact solution),
By using integrating factor(IF) $e^{\int f(x)dx}$
$y'(x) + f(x)y(x) = g(x)$
$e^{\int -6\,dx} = e^{-6x}$
$e^{-6x}y'(x) - 6e^{-6x}y(x) = e^{-6x}$
$\frac{d}{dx}e^{-6x}y = e^{-6x}$
$\int \frac{d}{dx}e^{-6x}y = \int e^{-6x}\,dx$
$e^{-6x}y(x) = -\frac{1}{6}e^{-6x} + c$
$y = 1$ When $x = 0$
$1(1) = -\frac{1}{6} + c$
$c = \frac{7}{6}$
$\therefore y(x) = \frac{7}{6}\,e^{2x} - \frac{1}{6}$

Using Kharrat-Toma for example 2,
$y' = 6y + 1$
$y' - 6y = 1$
$B[y' - 6y] = B[1]$
$B[y'] - 6B[y] = B[1]$

Applying the kharrat-Toma transform on (1), we get

$$\frac{1}{s^2}G(S) - s^3y(0) - 6G(s) = s^5$$
$$\frac{1}{s^2}G(S) - s^3 - 6G(s) = s^5$$
$$G(S)\left[\frac{1}{s^2} - 6\right] = s^5 + s^3$$
$$G(S) = \frac{s^7 + s^5}{1 - 6s^2}$$
$$= \frac{s^7}{1 - 6s^2} + \frac{s^5}{1 - 6s^2}$$
$$= \frac{s^5}{1} - \frac{s^5 - 7s^7}{1 - 6s^2} + \frac{s^5}{1 - 6s^2}$$
$$= -\frac{1}{6}s^5 + \frac{7}{6}\frac{s^5}{(1 - 6s^2)}$$
$$= B\left[-\frac{1}{6}\right] + B\left[\frac{7}{6}e^{6x}\right]$$

Applying the inverse Kharrat-Toma transform then the exact solution for the Initial value problem is

$$y(x) = -\frac{1}{6} + \frac{7}{6}\,e^{6x}$$

Example 3:
We consider the initial value problem
$$\begin{cases} y' = -3y + 5 \\ y(0) = 1 \end{cases}$$
(2)

Using analytical approach (Exact solution),
By using integrating factor(IF) $e^{\int f(x)dx}$
$y'(x) + f(x)y(x) = g(x)$
$e^{\int 3dx} = e^{3x}$
$e^{3x}y'(x) + 3e^{3x}y(x) = 5e^{3x}$
$\frac{d}{dx}e^{3x}y = 5e^{3x}$
$\int \frac{d}{dx}e^{3x}y = \int 5e^{3x}\,dx$
$e^{3x}y(x) = \frac{5}{3}e^{3x} + c$
$y = 1$ When $x = 0$
$1(1) = \frac{5}{3} + c$
$c = -\frac{2}{3}$

$\therefore y(x) = -\frac{2}{3}e^{-3x} + \frac{5}{3}$

Using Kharrat-Toma for example 2,

$y' = -3y + 5$

$y' + 3y = 5$

$B[y' + 3y] = B[5]$

$B[y'] + 3B[y] = B[5]$

Applying the kharrat-Toma transform on (1), we get

$$\frac{1}{s^2}G(S) - s^3y(0) + 3G(s) = 5s^5$$

$$\frac{1}{s^2}G(S) - s^3 + 3G(s) = 5s^5$$

$$G(S)\left[\frac{1}{s^2} + 3\right] = 5s^5 + s^3$$

$$G(S) = \frac{5s^7 + s^5}{1 + 3s^2}$$

$$= \frac{5s^7}{1 + 3s^2} + \frac{s^5}{1 + 3s^2}$$

$$= \frac{s^5}{1} - \frac{s^5 - 2s^7}{1 + 3s^2} + \frac{s^5}{1 + 3s^2}$$

$$= \frac{5}{3}s^5 - \frac{2}{3}\frac{s^5}{(1 + 3s^2)}$$

$$= B\left[\frac{5}{3}\right] - B\left[\frac{2}{3}e^{-3x}\right]$$

Applying the inverse Kharrat-Toma transform then the exact solution for the Initial value problem is

$$y(x) = \frac{5}{3} - \frac{2}{3}e^{-3x}$$

II. CONCLUSION

The main aim of this paper is to present some fundamental properties of newly defined integral transform "Kharrat-Toma Transform" as proposed by Kharrat et al in [24] . Convolution theorem is also proved for Kharrat-Toma transform. It provides a new mathematical tool to solve ordinary differential equations of variable and constant coefficients with initial condition. We validated the new integral transform method by giving a sophisticated analytical solution to solve ordinary differential equations.

FUTURE WORK

We are already improving on the existing method for its ability to solve non-linear differential equations and the existence of it's orthogonality property which was not mentioned by [24]

REFERENCES

[1]. Laplace, P.S., 1820. ThÂt'eorie Analytique des ProbabilitiÂt'es, Lerch, Paris, 1(2).

[2]. Stieltjes, T.S., 1894. Recherches sur les fractions continues, Annales de la Faculte des Sciences de Toulouse, pp: 1-123.

[3]. Mellin, H., 1896. Â´lUber die fundamentale Wichtgkeit des Satzes von Cauchy fÂ´lur die Theorien der Gammaund der hypergeometrischen funktionen, Acta Soc. Fennicae, 21: 1-115.

[4]. La Fourier, J., 1822. ThÂt'eorie Analytique de la Chaleur, English Translation by A. Freeman, Dover Publications.

[5]. Hilbert, D., 1912. GrundzÂ´luge einer allgemeinen Theorie der linearen Integralgeichungen, pp: 75-77.

[6]. Radon, J., 1917. Â´lUber die Bestimmung von Funktionen durch ihre Integralw Â´larte lÂ´langs gewisser manningfaltigkreiten, Ber Verh. Akad. Wiss. Math. - Nat., Leipzig, 69: 262-277.

[7]. Debnath, L., 1960. On Laguerre transform, Bull. Calcutta Math. Soc., 55: 69-77.

[8]. Watugula, G.K., 1993. Sumudu transform: a new integral transform to solve differential equations and control engineering problems, International Journal of Mathematical Education in Science and Technology, 24(1): 35-43.

[9]. Khan Z.H. and W.A. Khan, 2008. Natural transform-properties and applications, NUST Journal of Engineering Sciences, 1: 127-133.

[10]. Elzaki, T.M., 2011. The new integral transform "Elzaki transform", Global Journal of Pure and Applied Mathematics, 7(1): 57-64.

[11]. Aboodh, K.S., 2013. The new integral transform "Aboodh Transform" Global Journal of Pure and Applied Mathematics, 9(1): 35-43.

[12]. Srivastava, H.M., 2015. A new integral transform and its applications, Acta Mathematica Scientia, 35B: 1386-1400.

[13]. Zafar, Z., 2016. ZZ transform method, International Journal of Advanced Engineering and Global Technology, 04(01).

[14]. Kamal, A. and H. Sedeeg, 2016. The new integral transform "Kamal Transform" Advances in Theoretical and Applied Mathematics, 11(4): 451-458.

[15]. Xiao-Jun, Y., 2016. A New Integral Transform Method For Solving Steady Heat-Transfer Problem, Thermal Science, Vol. 20, Suppl., 3: S639-S642.

[16]. Mohand M.A. Mahgoub, 2017. The new integral transform "Mohand Transform" Advances in Theoretical and Applied Mathematics, 12(2): 113-120.

[17]. Ahmadi, S.A.P., H., Hosseinzadeh, A.Y.A. Cherati, 2019. New Integral Transform for Solving Higher Order Linear Ordinary Laguerre and Hermite Differential Equations. Int. J. Appl. Comput. Math, Springer 5-142.

[18]. Saadeh, R., A. Qazza and A. Burgan, 2020. A new Integral Transform: ARA Transform and Its Properties and Applications. Symmetry. MDPI, 12: 935.

[19]. Kharrat, B.N. and G. Toma, 2019. Differential TransformMethod For Solving Initial Value Problems

Represented By Strongly Nonlinear Ordinary Differential Equations. Middle-East Journal of Scientific Research, 27(7): 576-579.
[20]. Kharrat, B.N. and G. Toma, 2019. Differential Transform Method For Solving Initial and Boundary Value Problems Represented By Linear and Nonlinear Ordinary Differential Equations of 14^{th} Order. World Applied Sciences Journal, 37(6): 481-485.
[21]. Kharrat, B.N. and G. Toma, 2020. Differential Transform Method for Solving Boundary Value Problems Represented by System of Ordinary Differential Equations of 4^{th} Order. World Applied Sciences Journal, 38(1): 25-31.
[22]. Kharrat, B.N. and G. Toma, 2018. Modified Homotopy Perturbation Method by Using Sumudu Transform for Solving Initial Value Problems Represented By System of Nonlinear Partial Differential Equations. World Applied Sciences Journal, 36(7): 844-849.
[23]. Kharrat, B.N. and G. Toma, 2020. Numerical Solution of Van Der Pol Oscillator Problem Using a New Hybrid Method. World Applied Sciences Journal, 38(4): 360-364.
[24]. Kharrat, Bachir & Toma, George. (2020). A New Integral Transform: Kharrat-Toma Transform and Its Properties. World Applied Sciences Journal. 436-443. 10.5829/idosi.wasj.2020.436.443.