ISSN No:-2456-2165

The Quantum World and the Realm of Fractional Calculus

Fractional order Quantum Particle Search Optimization

Ansh Sarkar (Author)
B. Tech Computer Science
Kalinga Institute of Industrial Technology
Bhubaneswar , India

Anubhab Swain (Author)
B. Tech Computer Science
Kalinga Institute of Industrial Technology
Bhubaneswar , India

Abstract:- Quantum mechanics emerged as a result of the scientific enquiry into the wave nature of light during the mid-17th century under the contribution of eminent scientists including Christian Huygens, Robert Hooke and James Maxwell. This was followed by the Young's Double Slit experiment [1] and coining of the term Quanta (small energy packets) by Max Planck in the Planck's hypothesis for emission and absorption of radiation or energy. Several other contributions ranging from Wien's law and Maxwell's equations of Electromagnetic radiation to Schrödinger's [3] and Klein-Gordon's equations [4] have resulted in the continuous development of this seemingly reality defying field of modern science.

$$\widehat{\boldsymbol{H}}|\boldsymbol{\psi_n(t)}\rangle = \boldsymbol{i\hbar}\frac{\boldsymbol{\partial}|\boldsymbol{\psi_n(t)}\rangle}{\boldsymbol{\partial t}}$$

$$\boldsymbol{Schrödinger\ Equation}\ [\mathbf{3}]$$

$$\frac{\mathbf{1}}{\boldsymbol{c^2}}\frac{\boldsymbol{\partial^2\emptyset_n}}{\boldsymbol{\partial t^2}} - \boldsymbol{\nabla^2\emptyset_n} + \left(\frac{\boldsymbol{mc}}{\boldsymbol{\hbar}}\right)^{\mathbf{2}}\boldsymbol{\emptyset_n} = \mathbf{0}$$

$$\boldsymbol{Klein-Gordon\ Equation}\ [\mathbf{4}]$$

As with most of the other fields in science, quantum mechanics has infiltrated other fields of scientific interest including but not limited to nano-materials, possible teleportation and not to mention, quantum computing [12]. The main focus of this chapter revolves around the optimization techniques, their quantum implementations and how they can be made more efficient by introducing a slight amount of fractional order calculus which introduces the long term memory required for faster convergence to optimal values. Most of the concepts required to understand this text have been explained in the subsequent sections. Readers should feel free to refer to the reference/bibliography section for further reading.

Keywords:- *Fractional Calculus [5] , Particle Swarm [27] Optimization [6] , Quantum Computing [12], Optimization tools and Techniques , Review*

I. INTRODUCTION

The main concern of engineering problems has always been about maximizing yields or minimizing losses. As science and technology develop, the complexity of optimization problems increases. Engineering problems in the areas of energy conversion, conservation and distribution, mechanical design, logistics, all the way to the reload of nuclear reactors often find in themselves an inherent need for optimization techniques .

The world of quantum mechanics is known for its unforgiving use of seemingly complex terminologies and mathematical equations, mostly defining concepts which seem to defy our perception of reality. The following section introduces the reader to a few of the algorithms [23] and underlying mathematical theorems which can prove useful for understanding the application of Fractional order differential equations (or fractional order calculus in general) and the concept behind Optimization Techniques , particularly the Fractional Order Quantum Particle Swarm Optimization technique (FQPSO) [27].

II. PARTICLE SWARM OPTIMIZATION [27]

The Particle Swarm Optimization (PSO) algorithm [6] is a metaheuristic algorithm, originally proposed by Kennedy and Eberhart, which is capable of solving various complex engineering problems using the power of swarm intelligence

Pseudo Code:

1. Initialising each particle
1.1. Initialise each X_i randomly
1.2. Initialize each V_i randomly
1.3. Evaluate the fitness of each X_i or : $f(X_i)$
1.4. Initialising $pbest_i$ with a copy of X_i
2. Initialize global best or $gbest$ with a copy of X_i
3. Repeat the following until stopping criterion achieved :
3.1. For every particle i:
3.1.1. Update V_i^t and X_i^t according to the formulas :

$$V_{ij}^{t+1} = wV_{ij}^t + c_1\left(pbest_{ij} - X_{ij}^t\right) + c_2 r_2^t(gbest_{ij} - X_{ij}^t)$$

And,
$$X_{ij}^{t+1} = X_{ij}^t + V_{ij}^{t+1}$$

ISSN No:-2456-2165

3.1.2. Evaluate the fitness $f(X_i^t)$
3.1.3. $pbest_i \leftarrow X_i^t$ *if* $f(pbest_i) < f(X_i^t)$
3.1.4. $gbest \leftarrow X_i^t$ *if* $f(gbest) < f(X_i^t)$

* (Please refer to the following section for the meanings of the symbols used in the above pseudo-code. The following section provides an in depth analysis of the PSO algorithm [22] and its working) The fitness function (also referred to as the objective function)

$$f(X) : R^D \rightarrow R$$

On a D-dimensional domain R^D represents the function which we are interested in minimising.

Particle Swarm Optimization [6] was originally intended to be used for simulating social behaviour such as the movement of organisms in swarms, for example: a flock of birds or a school of fishes. New and more complex forms/variants of the PSO algorithm [23] are being continuously developed for improving [18] the optimizational efficiency of these algorithms. The new variants include: improved genetic algorithms and Accelerated Particle Swarm Optimization Techniques.

Since the Quantum Particle Swarm Optimization Algorithm [8] as well as the Fractional Quantum Particle Swarm Optimization Algorithm [9] are both based on solid foundations of the PSO concept, we shall be dedicating an entire section only to contribute to our understanding of the fundamentals of the PSO technique.

A. Attractors

The local attractor (better known as the private guide) [7] of a particle can be defined as the best position with respect to f (the objective function) that particle has encountered so far. The swarm [26] also shares a standard memory also called the worldwide attractor (better referred to as the local guide), which is the best position any of the particles within the entire population has found so far. The movement of the entire swarm is hence governed by the local attractor [7] , the global attractor and the movement equations (which shall be discussed in the upcoming section).

B. Understanding the Particle Swarm Optimization Algorithm

There are several approaches that one could perform to maximize or minimize a function in order to find the optimum value. Though there exists a wide range of optimization algorithms that could be used, there is not a particular one that is considered to be the best for any case. An optimization method that works fine for one problem might not be so for another one; it depends on several features, for example, whether the function is concave or convex, and whether it is differentiable or not. To solve such a problem, an individual must understand various optimization techniques so as to select the algorithm that best fits on the optimization problems parameters.

One such optimization algorithm is Particle Swarm Optimization (PSO) [6] which was developed in 1995 by the authors Kennedy and Eberhart, inspired by the behaviour of social organisms in groups, such as bird and fish schooling or ant colonies. This algorithm mimics the interaction between members of a particular group to share information. It has been applied to various areas of optimization and in combination with other existing algorithms. This algorithm performs the look for the optimal solution through agents, also mentioned as particles, whose trajectories are adjusted by a deterministic and a stochastic component. Each and every particle is influenced by its 'best' (local attractor [7]) achieved position and the group's 'best' position (global attractor), but tends to move randomly.

This algorithm also has a competitive performance with the Evolutionary Programming (EP) [29], Genetic Algorithm (GA), Evolution Strategies (ES), Genetic Programming (GP) and many other classic algorithms [23].

The goal of an optimization problem is to determine a variable represented by a vector $X = [x_1\ x_2\ x_3\ \dots\ x_n]$, that minimizes or maximizes depending on the proposed optimization formulation of the function $f(x)$. The variable vector X is known as position vector; this vector represents a variable model and it is n dimensional vector, where n represents the number of variables that may be determined in a problem, that is, the latitude and therefore the longitude within the problem of determining some extent to land by a flock. On the other hand, the function $f(x)$ is referred to as the fitness or objective function, which assesses how good or bad a position X is, i.e. how good a particular landing point a bird thinks it is after the animal finds it, evaluation of which is subjected to several survival criteria.

Consider a swarm constituting P particles, in such a case we know that there exists a position vector $X_i^t = (x_{i1}x_{i2}x_{i3}\dots x_{in})^T$ and a velocity vector $V_i^t = (v_{i1}v_{i2}v_{i3}\dots v_{in})^T$ at t iteration for each one of the i particle that composes it. These vectors are updated through the dimension j according to the following equations:

$$V_{ij}^{t+1} = wV_{ij}^{t+1} + c_1r_1^t(pbest_{ij} - X_{ij}^t) + c_2r_2^t(gbest_j - X_{ij}^t) \quad \text{Eq.(1)}$$

and

$$X_{ij}^{t+1} = X_{ij}^t + V_{ij}^{t+1} \quad \text{Eq.(2)}$$

where $i = 1,2,3\dots P$ and $j = 1,2,3\dots n$.

Eq. (1) denotes the presence of three different contributors to the particle's movement in an iteration, giving rise to three terms which shall be further discussed. On the other hand, Eq. (2) updates the particle's positions. The inertia weight constant represented as w is a positive constant value in case of classical Particle Swarm [26] Optimization and is important for balancing the global search, also referred to as exploration (in case higher values are set), and local search, known as exploitation (in case lower values are set).This is one of the main differences between the classical and the derived versions of PSO.

ISSN No:-2456-2165

The first term of the velocity update equation is given by the product between parameter w and the particle's previous velocity, which is how it takes into consideration the particle's previous motion while calculating the current one. Hence, for example, if $w = 1$, the particle's motion is completely influenced by its previous motion and therefore the particle may keep going on , in the same direction whereas if $0 \leq w < 1$, a partial influence is imparted , which implies that a particle wanders and goes to other regions in the search domain. Therefore, a reduction in the inertia weight parameter makes the swarm explore more areas in the searching domain, thus enhancing the chances of finding a global optimum. However, using lower w values, comes with the price of an increased simulation time as the swarm takes a longer time to converge.

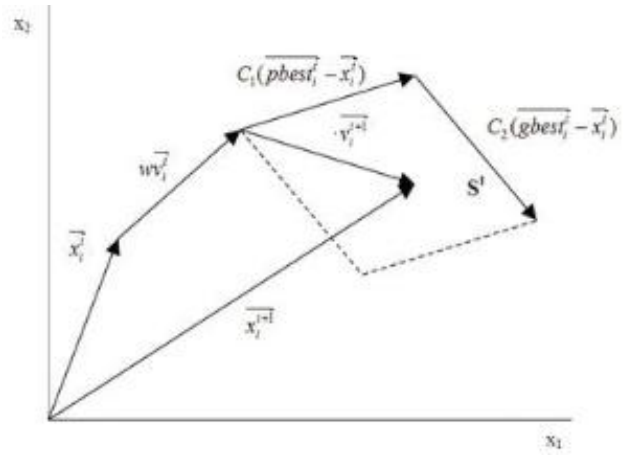

Fig 1

The individual cognition term, (indicated by the 2nd term in Eq.(1)) , is calculated using the difference between the particle's own best position, for example, $pbest_{ij}$, and its current position X_{ij}^t. One may notice that the idea behind this term is that as the particle gets more distant from the $pbest_{ij}$ position, the difference $(pbest_{ij} - X_{ij}^t)$ must increase; therefore, this term increases, attracting the particle to its own best position. The parameter c_1 exists as a product in this term while being a positive constant and an individual-cognition parameter, helping particles to weigh the importance of their own previous experiences. The second parameter that is made up of the product of the second term is r_1, and this is a random value parameter lying within the range [0,1] . This random parameter helps in avoiding premature convergences, thereby increasing the chances at finding the most likely global optima.

Lastly , the third term in the equation is responsible for the social learning behaviour of the swarm. It makes it possible for all the particles in the swarm to be able to share the information of the best point achieved with each other , regardless of which particle had found it, for example, $gbest_j$. It has the same format as that of the term corresponding to the individual learning property of the swarm. Hence : $(gbest_j - X_{ij}^t)$ is the difference that acts as an attraction for the particles to the best point until it is found at some t iteration. In the same way , the variable c_2 is a parameter involved in social learning , which weighs the importance of the global learning of the swarm while r_2 plays the exact same role as r_1.

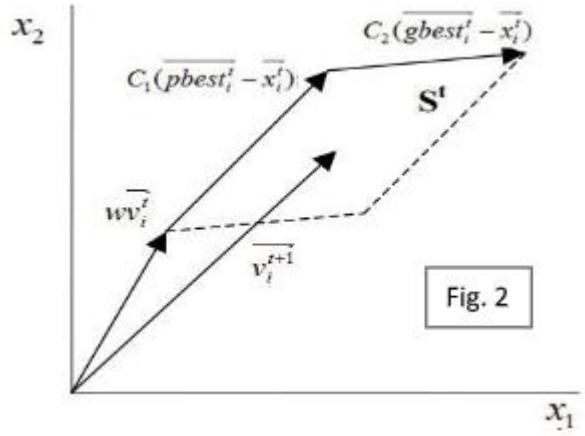

Fig 2

III. UNDERSTANDING QUANTUM COMPUTERS AND THE QUANTUM WORLD

A. Quantum Physics

The emergence of Quantum mechanics in the early 1900s was mainly driven by the urge to develop a framework to explain nature on the nanoscopic scale of atoms and drive advancements in areas such as transistor technology, efficient or better laser systems, and more accurate high resolution magnetic resonance imaging. The merging of quantum mechanics and information theory was a debate in and around the 1970s but garnered little attention. That was until 1982, when physicist Richard Feynman gave a talk revolving around the fact that computing based on classical logic could not process calculations describing quantum phenomena. On the other hand, quantum phenomena based computing, configured to simulate and test other quantum phenomena, could theoretically act as a great solution as it would not be subject to the same bottlenecks as the ones faced by modern day classical computational systems. This application of quantum computing [12] eventually gave rise to the field of quantum simulation but failed to spark much research activity or interest at the time.

This lack of interest in quantum computing [12] and research related to the field however took a turn in the year 1994, when mathematician Peter Shor developed a quantum algorithm, capable of finding the prime factors of large numbers efficiently. The word, "efficiently" here refers to a time range of practical relevance, which is currently beyond the capability of even state-of-the-art classical algorithms.

B. Quantum Computing

The fundamental motivation behind both Quantum and classical computers remains the same , i.e. to help and try to solve problems in an efficient manner within a particular time frame, but at their core , the way they manipulate the data to derive answers is almost completely different. In the following parts of this section we aim to provide an

ISSN No:-2456-2165

explanation targeting the uniqueness of quantum computing [12] systems by introducing the two fundamental underlying principles of quantum mechanics crucial for their operation namely : superposition and entanglement [21].

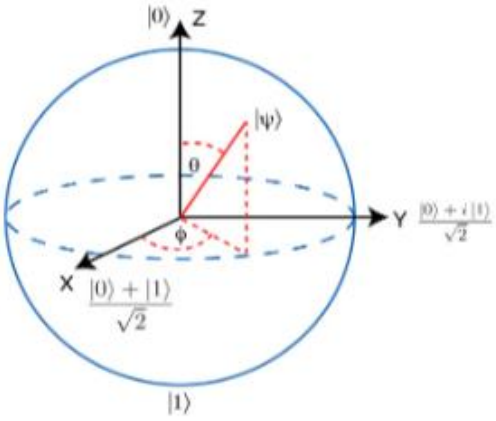

Fig 3

C. Need for Quantum Computing

The possibility of someday developing a quantum computer sophisticated and capable enough to pull off feats such as executing Shor's algorithm for large numbers which might effectively result in a revamp of currently implemented cryptographic tools and systems , has been one of the motivators for advancing the field of quantum computation. It is however important to understand that quantum computers are likely to deliver tremendous speed-ups only for specific types of problems which can be simulated in the quantum world. In other words , quantum computing [12] does not mean faster computers , it just means that we shall be able to solve specific problems , within a reasonable time-frame compared to what would have required traditional or classical computers years to compute. Researchers are working towards enhancing their understanding of which problems are best suited for quantum speed-ups and focusing on the development of algorithms to demonstrate them. Scientists working in the field of quantum computing [12] currently involved in active research believe that quantum computers should be able to help immensely with problems related to optimization of parameters or resources , which can play key roles in advancements in fields ranging from everything from defence to financial trading.

There are additional applications focused at exploiting qubit based systems for fields that are not related to computing or simulation and are active areas of research. Research is currently underway on two of the most prominent application areas or use cases of qubit based computational systems namely :

1. **Quantum Sensing and Metrology** , leveraging the extreme sensitivity of qubits to environmental factors in order to realize and improve sensing capabilities beyond the classical shot noise limit, and
2. **Quantum Networks and Communications** , which are thought to lead to revolutionary breakthroughs in the realm of information / data sharing .

D. The building blocks of Quantum Computers: Qubits

A well known fact is that modern day computers are binary digital systems which use bits - or in other words a stream of electrical or optical pulses representing the two states of either 1s or 0s , highs or lows and on or off. Everything in the digital world ranging from our tweets and emails, to our credit card information , songs and YouTube videos are nothing more than long strings of these 1s and 0s.

In quantum computing [12] systems the building blocks consist of a qubit or quantum bit which is also the basic unit of quantum information - it represents the quantum version of the classical binary bit usually physically designed and working as a two-state device. The design or structure of a qubit essentially represents a somewhat similar two-state (or two-level) quantum-mechanical system, often regarded as perhaps one of the simplest quantum systems having the ability to display the peculiarity of quantum mechanics. The similarity in representation of states in a qubit in comparison with a classical bit can be seen by taking the spin of an electron as an example in which the two levels can be considered as the two spins namely : spin up and the spin down of an electron ; or even the polarization of a single photon in which the states can be considered according to whether the light has undergone vertical polarization or horizontal polarization. The generation and maintenance of stable qubits [13] poses huge scientific and engineering challenges. Popular Tech Conglomerates, such as IBM, Google [19], and Rigetti Computing, make use of superconducting circuits which are cooled down to mind cryogenic temperatures , even colder than that in deep space. Other systems like IonQ (Ion Capture Technology) [14] work by trapping and isolating individual atoms with the help of strong electromagnetic fields which are produced by strong superconducting electromagnets on silicon chips kept inside ultra-high-vacuum chambers. Whatever be the methods used , the end goal is to isolate the qubits [13] in a controlled quantum state.

Unlike binary bits, qubits have some quirky quantum properties due to which a connected group of them can provide way more processing power than the same number of binary bits. The properties that enable the qubits [13] to exhibit such extraordinary performance are superposition and entanglement [21].

There exists 2 orthogonal z-basis states of a qubit which can be defined as follows :

- $|0\rangle$
- $|1\rangle$

The two orthogonal x-basis states are:

1. $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$
2. $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

ISSN No:-2456-2165

Similarly , the two orthogonal y-basis states of a qubit can be defined as :

1. $|R\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$
2. $|L\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$

The basis states can be visualized as points which are located at the opposite points on the Bloch sphere representation of the current state of a single qubit.

E. Understanding Superposition

The property of superposition is what sets apart a quantum bit from a classical bit is the fact that a qubit can be in superposition. Superposition is one of the most weird , intriguing and yet among the fundamental principles of quantum mechanics. We have read about the Fourier Series and its applications in signal processing. Using the same principle in classical physics, a wave describing a musical tone can therefore be seen as a combination or superposition of several waves with different frequencies that are added together. In the same manner , in case of quantum mechanics a superposed quantum state can be broken down or decomposed into a linear combination of a number of other distinct quantum states.

The superposition of the Qubits [13] can be broken down and represented as a linear combination of the basis states $|0\rangle$ and $|1\rangle$. When a qubit is measured (only observables are measurable), the qubit falls or collapses to one of its Eigen states which is then reflected in the measured value. For example, if a qubit exists in an equally weighted superposed state, measuring it or trying to observe it , would make it collapse to one of its basis states$|0\rangle$or$|1\rangle$ with an equal probability of 50%. The $|0\rangle$basis state on measurement or observation collapses and always gives the result as 0 and the basis state$|1\rangle$always converts or collapses to 1.

It should however be kept in mind that Quantum Superposition is fundamentally different from summation and superposition of classical waves. A *n* qubit containing quantum computer can in total exist in a superposition of 2^n states: from$|000...0\rangle$ to $|111...1\rangle$. But on the other hand , if we take the example of a classical situation such as playing *n* musical sounds with distinct frequencies, it can be observed that we can only obtain a superposition of *n* frequencies. Therefore , adding classical waves scales linearly, whereas the superposition of quantum states scales exponentially.

F. Quantum Entanglement: Spooky Action at a Distance [21]

Entanglement is one among the many counter-intuitive phenomena in the realm of Quantum Physics. Entanglement refers to a phenomenon when the quantum state of a pair or group of particles cannot be described independently of the quantum state of the other particle(s). Therefore the quantum system as a whole can be described as being in a definite state where the measurement of each particle affects every other particle, even though the individual parts of the system are not in a definite state.

The entanglement of two qubits [13] brings into existence a special connection between them. The effects of the entanglement become clear from the observations obtained after measurements. Upon measurement , the outcome of observing the individual qubits could be either 0 or 1. However , since the qubits are entangled with each other , the outcome of the measurement of a single qubit will always be correlated or influenced by the measurement on the other qubits [13]. This happens , even if the particles are far apart and separated from each other by large distances. The Bell state configuration of qubits [13] can be given as an example for the observation of the behaviour of an entangled qubit system.

As an example, let us consider two particles created in such a way that the total spin of the system is zero. A certain reference axis is considered and the spin of the particles is measured along that axis. Let us suppose that the spin of one of the generated particles measured with respect to a certain axis is found to be counter clockwise, then it can be guaranteed that a measurement of the spin of the other particle (with reference to the same axis) will exhibit a clockwise spin. It appears as if one of the entangled particles is able to feel or detect immediately that a measurement has been performed on the other entangled particle and accordingly locks what the outcome should be, but this is not the case. All this happens instantaneously, without the exchange of any form of information between the entangled particles even if they were located billions of miles away from each other , this entanglement would still exist.

IV. FRACTIONAL CALCULUS

Fractional calculus [5] as the name suggests, deals with the analysis of differentials of fractional order. The topic is in no way a recent development and has seen a lot of contribution from people from the field of pure mathematics. Only recently has it undergone major, heated development by the contribution of various eminent mathematicians from the field of applied mathematics. Nevertheless there is a wide variety of theorems to choose from in order to navigate the realm of Fractional Calculus [5] and they are usually not compatible (equivalent) to each other. Only in recent years have scientists around the world discovered the use of fractional calculus and its theories in understanding various natural frameworks ranging from viscoelastics, signal processing and of course now in Quantum Particle Swarm Optimization technique [8].

The use of Convolutional kernels for evaluating fractional integrals, The Riemann-Liouville Integral / Derivative, The Reisz fractional derivative , The Caputo fractional derivative and the Grünwald–Letnikov [2] derivative are some of the frameworks in place for understanding / evaluating fractional order differentials. The most popular among these: The Grünwald–Letnikov [2] derivative, has been discussed in further details in the upcoming sections.

ISSN No:-2456-2165

A few terms and functions to know about while discussing fractional calculus have been listed as follows .

A. Analytic Functions:

Functions which can be represented in the form of a power series are known as Analytic functions. Their importance lies in the fact that they encompass and contain most of the commonly encountered mathematical functions.

These functions are usually divided into two separate types : Real analytic functions and Complex analytic functions (also called holomorphic functions). This class of functions is closed relative to the fundamental operations of arithmetic which are used in algebra, analysis and the other fields of analytic functions and applied mathematics.

B. The Euler-Gamma function:

The Gamma function representable as $\Gamma(n)$ can be defined as an extended version of the factorial function (n!) which is otherwise, only limited to whole numbers. The gamma function hence extends the factorial function to include complex as well as fractional values of n.

It is related to the factorial of a number as: $\Gamma(n) = (n-1)!$ The Euler-Gamma function is analytic everywhere except at n = 0 , -1 , -2 , … and residue at $z = -i$ is given by :

$$Res_{z=-i}\,\Gamma(z) = \frac{(-1)^i}{i!}$$

C. Grünwald–Letnikov derivative

Just like modules enhance/extend the functionality of a programming language, various theorems in mathematics act as patches which help to increase the domain of functioning of a particular field of mathematics.

The Grünwald–Letnikov (GL) derivative [2] does the same for the field of differential calculus and hence extends the domain of differential equations in order to include fractional order differentials. It allows us to take the derivative of a function, non-integer number of times

D. Understanding / Deriving the Grünwald–Letnikov derivative [2]

We all are well aware that the differential of a function representable as $f'(x)$ is:

$$f'(x) = \lim_{c\to 0}\frac{f(x+c)-f(x)}{c}$$

The derivative can be applied recursively on a function to get derivatives of higher orders. So we can write the 2nd derivative as:

$$f''(x) = \lim_{c\to 0}\frac{f'(x+c)-f'(x)}{c}$$

$$= \lim_{c_1\to 0}\frac{\lim_{c_2\to 0}\frac{f(x+c_1+c_2)-f(x+c_1)}{c_2} - \lim_{c_2\to 0}\frac{f(x+c_2)-f(x)}{c_2}}{c_1}$$

Here we assume that the infinitesimally small (differential) element converges synchronously. This can be further verified by using the Mean Value Theorem or M.V.T. We now generalize this by using our knowledge of binomial expressions in order to obtain :

$$f^{(n)}(x) = \lim_{c\to 0}\frac{1}{c^n}\sum_{0\le m\le n}(-1)^m\binom{n}{m}f(x+(n-m)c)$$

We now remove the restriction that n should be a positive integer and hence derive the form:

$$D^q f(x) = \lim_{c\to 0}\frac{1}{c^q}\sum_{0\le m\le \infty}(-1)^m\binom{q}{m}f(x+(q-m)c)$$

This, is the general form of the Grünwald–Letnikov (GL) derivative [2].

The Grünwald–Letnikov (GL) derivative [2] :

$${}_a^{GL}D_x^{\alpha}f(x) = \frac{d^{\alpha}}{[d(x-a)]^{\alpha}}f(x) = \lim_{n\to\infty}\left\{\frac{\left(\frac{x-a}{n}\right)^{-\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{n-1}\frac{\Gamma(k-a)}{\Gamma(k+1)}f\left(x-k\left(\frac{x-a}{n}\right)\right)\right\}$$

Where, $f(x)$ is a differentio-integrable function, [a, x] is the domain of the function, Γ represents the Euler-Gamma function, α represents the fractional order of the derivative and ${}_a^{GL}D_x^{\alpha}f(x)$ represents the GL derivative operator.

When n becomes large enough, the limit symbol is neglected (as we then consider infinitesimal sections) . We can hence rewrite the above equation for large values of n as:

$$\frac{d^{\alpha}}{dx^{\alpha}}f(x) \cong \frac{x^{-\alpha}n^{\alpha}}{\Gamma(-\alpha)}\sum_{k=0}^{n-1}\frac{\Gamma(k-\alpha)}{\Gamma(k+1)}f\left(x - \frac{kx}{n}\right); \textit{for large } \boldsymbol{n}$$

The above equation can be further generalised for a 1D signal as follows (Note: the equation given below is not convenient for thorough analysis but is often used for numerical approximations):

$$\frac{d^{\alpha}}{dx^{\alpha}}f(x) \cong f(x) + (-\alpha)f(x-1) + \frac{(-\alpha)(-\alpha+1)}{2}f(x-2) + \cdots + \frac{(-\alpha)(-\alpha+1)\dots(-\alpha+n)}{n!}f(x-n)$$

ISSN No:-2456-2165

V. QUANTUM PARTICLE SWARM OPTIMIZATION

The major/significant difference between classic PSO and Quantum PSO is the use of the wave function in QPSO in order to facilitate optimization or determine the position of a given particle in 3D space.

Trajectory analysis according to [citation] demonstrated that each particle in practice converges to its corresponding local attractor [7] or C_{id} which can be demonstrated by the following equation:

$$C_{id}(t) = a.pbest_{id}(t) + (1-a).gbest_{id}(t)\,, a{\sim}U(0,1)$$

Where, $a = \frac{c_1 r_1}{(c_1 r_1 + c_2 r_2)}$

The local attractor [7] is thus a randomly distribution or pattern that may be analyzed statistically but may not be predicted precisely or in other words a stochastic attractor of the i^{th} particle that lies inside a hyper-rectangle with $pbest_{id}$ and $gbest_{id}$ as the two ends of its diagonal.

Sun et al. studied the convergence behaviour of PSO and proposed a novel PSO model from quantum mechanics abbreviated as QPSO or the Quantum Particle Swarm Optimization [8]. Based on the Delta potential, the quantum behaviour of particles are considered. In the framework of quantum time-space, the quantum state of a particle is represented by a wave function $\psi(x,t)$.
In 3D space, $\psi(x,t)$ is given as

$$|\psi|^2 dxdydz = Qdxdydz$$

where Q is the probability of finding the particle at a particular point in space (with respect to the xyz coordinate system).Here , $|\psi|^2$ represents the probability density of the particle.

As a probability density function, we have:

$$\int_{-\infty}^{+\infty} |\psi|^2 dxdydz = \int_{-\infty}^{+\infty} Qdxdydz = 1$$

The normalized version of ψ can be represented as:

$$\|\psi\| = \psi(y) = \frac{1}{\sqrt{L}} e^{\frac{-|y|}{L}}$$

As a result, Q and the corresponding distribution function *F* can be obtained as:

$$Q(y) = |\psi(y)|^2 = \frac{1}{L} e^{\frac{-2|y|}{L}}$$

And :

$$F\big(X_{id}(t+1)\big) = 1 - e^{-\frac{2|p_{id}(t) - X_{id}(t+1)|}{L_{id}(t)}}$$

Where $L_{id}(t)$ denotes the standard deviation, which describes the search space of each particle. The Monte Carlo method can now be used in order to obtain the position of the particle as follows :

$$s = \frac{1}{L}u = \frac{1}{L} e^{-\frac{2|y|}{L}}, u \sim (0,1) and\ s \sim \left(0, \frac{1}{L}\right)$$

Then by comparing:

$$u = e^{-\frac{2|y|}{L}}$$

Let $y = x - c$, we have:

$$x = c \pm \frac{L}{2} \ln\left(\frac{1}{u}\right)$$

The convergence condition for PSO is given by:

$$x \to c\,, when\ t \to \infty$$

Let *L* be a function of time, then for the above condition to hold true, we have:

$$L = L(t)\,, L \to 0\,, when\ t \to 0$$

We can further write the iterative version for the i^{th} particle as:

$$X_{id}(t+1) = C_{id} \pm \frac{L_{id}}{2} \ln\left(\frac{1}{u}\right)$$

A global point called mean best position is introduced to evaluate the value of $L_{id}(t)$. This global point, which is denoted by *mbest* and can be computed as the mean of the *pbest* positions of all the particles present in the swarm, which can be given s:

$$mbest(t) = (mbest_1(t), mbest_2(t), \dots, mbest_d(t)$$
$$= \frac{1}{n}\sum_{i=1}^{n} p_{i1}(t), \frac{1}{n}\sum_{i=1}^{n} p_{i2}(t), \dots, \frac{1}{n}\sum_{i=1}^{n} p_{id}(t),$$

The values of $L_{id}(t)$ is calculated by:

$$L_{id}(t) = 2\beta |m_d(t) - X_{id}(t)|$$

And finally, the position of the particle X_{id} can be given by:

$$X_{id}(t+1) = C_{id}(t+1) \pm \beta |mbest_d - X_{id}(t)| \ln\left(\frac{1}{u}\right)$$

Where parameter β is step size, which is utilized to control the convergence speed. U(0,1) is the shorthand for Standard Uniform Random Variable Distribution in between 0 and 1.

A. QPSO with the fractional-order position : FQPSO

ISSN No:-2456-2165

Fractional Calculus [5] is often regarded to have excellent long term memory characteristics. From the definition of Grünwald-Letnikov differential [2] (under fractional calculus sub-section), it can be seen that the fractional derivative is computed with the previous/historical states which makes it suitable for the iterative procedure of intelligent optimization algorithms like the Fractional Quantum Particle Swarm Optimization algorithm [9] , which introduces fractional index based swarm optimization.

To further improve the speed and accuracy of convergence of QPSO algorithm/technique, in this section, the proposed QPSO with the fractional-order position is detailed. Initially, the original position is rearranged to modify the order of the position derivative, which can be derived as follows:

Equations corresponding to all possible relations between rand and $\mathrm{mbest_{id}}$. (rand~U(0,1))

$\boldsymbol{Case\ 1:}\ rand > 0.5\ , mbest_d > X_{id}(t)$

$$X_{id}(t+1) = C_{id}(t) + \beta.\ln\left(\frac{1}{u}\right).\left(mbest_d - X_{id}(t)\right)$$

$$and\ ,\quad rand > 0.5\ , mbest_d > X_{id}(t)$$

$\boldsymbol{Case\ 2:}\ rand > 0.5\ , mbest_d < X_{id}(t)$

$$X_{id}(t+1) = C_{id}(t) + \beta.\ln\left(\frac{1}{u}\right).\left(X_{id}(t) - mbest_d\right)$$

$$and\ rand > 0.5\ , mbest_d < X_{id}(t)$$

$\boldsymbol{Case\ 3:}\ rand < 0.5\ , mbest_d > X_{id}(t)$

$$X_{id}(t+1) = C_{id}(t) - \beta.\ln\left(\frac{1}{u}\right).\left(mbest_d - X_{id}(t)\right)$$

$$and\ rand < 0.5\ , mbest_d > X_{id}(t)$$

$\boldsymbol{Case\ 4:}\ rand < 0.5\ , mbest_d < X_{id}(t)$

$$X_{id}(t+1) = C_{id}(t) - \beta.\ln\left(\frac{1}{u}\right).\left(X_{id}(t) - mbest_d\right)$$

$$and\ rand < 0.5\ , mbest_d < X_{id}(t)$$

All the above equations can now be uniformly written as :

$$X_{id}(t+1) - X_{id}(t) = C_{id}(t) + \beta.\ln\left(\frac{1}{u}\right).(mbest_d) - \left(\beta.\ln\left(\frac{1}{u}\right) \pm 1\right)X_{id}(t)$$

The left side of the above equation is the discrete version of the derivative with $\alpha = 1$. We can hence extend / generalize the above equation to get a general fractional order differential as follows:

$$D^{\alpha}\left(X_{id}(t+1)\right) = C_{id}(t) + \beta.\ln\left(\frac{1}{u}\right).mbest_d - \left(\beta.\ln\left(\frac{1}{u}\right) \pm 1\right)X_{id}(t) \quad \ldots\ eq.1$$

$$when\ rand > 0.5\ , mbest_d > X_{id}(t)\ or\ ,$$

$$when\ rand < 0.5\ and\ mbest_d < X_{id}(t)$$

And ,

$$D^{\alpha}\left(X_{id}(t+1)\right) = C_{id}(t) - \beta.\ln\left(\frac{1}{u}\right).mbest_d + \left(\beta.\ln\left(\frac{1}{u}\right) \pm 1\right)X_{id}(t) \quad \ldots\ eq.2$$

$$when\ rand > 0.5\ , mbest_d < X_{id}(t)\ or\ ,$$

$$when\ rand < 0.5\ and\ mbest_d > X_{id}(t)$$

Researchers have demonstrated the fact that for $\sim U(0,1)$, smoother variations and a prolonged memory effect can be introduced which might possibly lead to a better accuracy than then the integral-order based model

To study the behaviour and accuracy of the proposed fractional-order strategy, a set of functions are tested with $\alpha \sim U(0,1)$ and $\Delta\alpha = 0.1$ which denotes the step size. To simplify the computational complexity, we usually truncate the simplified form of the Grünwald-Letnikov differential [2] and only use the first four terms, so we have:

$$D^{\alpha}\left(X_{id}(t+1)\right) = X_{id}(t+1) - \alpha X_{id}(t) - \frac{1}{2}\alpha(1-\alpha)X_{id}(t-1) - \frac{1}{6}\alpha(1-\alpha)(2-\alpha)X_{id}(t-2) - \frac{1}{24}\alpha(1-\alpha)(2-\alpha)(3-\alpha)X_{id}(t-3)$$

$$\ldots truncated\ to\ 4\ terms$$

Using the above result we can redefine / modify eq.1 as :

$$\left(X_{id}(t+1)\right) = C_{id}(t) + \beta.\ln\left(\frac{1}{u}\right).mbest_d - \left(\beta.\ln\left(\frac{1}{u}\right) \pm 1 - \alpha\right)X_{id}(t) + XX_{id}(t)$$

We also modify eq.2 as :

$$\left(X_{id}(t+1)\right) = C_{id}(t) - \beta.\ln\left(\frac{1}{u}\right).mbest_d + \left(\beta.\ln\left(\frac{1}{u}\right) \pm 1 + \alpha\right)X_{id}(t) + XX_{id}(t)$$

Where $XX_{id}(t)$ is defined as :

$$XX_{id}(t) = \frac{1}{2}\alpha(1-\alpha)X_{id}(t-1) + \frac{1}{6}\alpha(1-\alpha)(2-\alpha)X_{id}(t-2) + \frac{1}{24}\alpha(1-\alpha)(2-\alpha)(3-\alpha)X_{id}(t-3)$$

ISSN No:-2456-2165

With respect to the position updating equations (equations of motion) of the particles it can be seen that the position of the particles depend not only on the position of the previous particle but also on the historical position of the particle itself at different points in time. The position acquired by the particle in subsequent steps or epochs is hence the result of it's long term memory. This long term memory has the ability to protect the population density (distribution) and the diversity to a certain extent.

B. Pseudo Code (Algorithm)

1. Initialising the FQPSO parameters (α , t , β , u)
2. Initialise the swarm of particles
3. For each particle i in population:

3.1. Computer $f(X_i)$, where $f(\,)$ represents the fitness function

3.1.1. If $f(X_i) < f(pbest_i)$:

$$pbest_i = X_i$$

End

3.1.2. If $f(pbest_i) < f(gbest)$:

$$gbest = pbest_i$$

End

3.2. Calculate Q using the equations

3.2.1. If

$$\left(rand > 0.5 \; and \; mbest_d < X_{id}(t)\right) or \left(rand < 0.5 \; and \; mbest_d > X_{id}(t)\right)$$

:

$$\left(X_{id}(t+1)\right) = C_{id}(t) + \beta.\ln\left(\frac{1}{u}\right).mbest_d - \left(\beta.\ln\left(\frac{1}{u}\right) \pm 1 - \alpha\right) X_{id}(t) + XX_{id}(t)$$

Else

If

$$\left(rand > 0.5 \; and \; mbest_d > X_{id}(t)\right)$$

or $\left(rand < 0.5 \; and \; mbest_d < X_{id}(t)\right)$

$$\left(X_{id}(t+1)\right) = C_{id}(t) - \beta.\ln\left(\frac{1}{u}\right).mbest_d + \left(\beta.\ln\left(\frac{1}{u}\right) \pm 1 + \alpha\right) X_{id}(t) + XX_{id}(t)$$

End

3.3. $t = t + 1$

4. Continue **Until** termination criterion is satisfied.

Note : Other algorithms like the Fractional order Darwinian Particle Swarm Optimization [28] also use Fractional Calculus [5] but since they belong to the category of evolutionary algorithm [29], we will not be covering it here. The reader may feel free to read the articles or resources given in the reference section. You can also refer to references section for experimental evidence of the efficiency of FQPSO over QPSO.

VI. APPLICATIONS OF THE ABOVE CONCEPTS AND (F)QPSO IN THE MODERN WORLD

A. Applications of Quantum Computing [17]

- As the new technologies have penetrated almost every aspect of our lives, **artificial intelligence** and machine learning have become some of the prominent areas right now. Some of the widespread applications that we see every day are in recognition of image, voice and handwriting samples. However, it becomes a challenging task for traditional computers, to match up the accuracy and speed as the number of applications have increased. And, that's where quantum computers are better than traditional computers as they can help in processing through complex tasks in very less time, which would have taken traditional computers thousands of years.

- **Weather forecasting** is another such field which includes several variables to consider, such as temperature, air pressure and air density, which makes it tough for it to be predicted accurately. Applications of quantum machine learning can help us improve our ability to recognize patterns, which will make it easier for scientists and researchers to predict harsh weather conditions and potentially save thousands of lives. Meteorologists will be able to analyse more detailed climate models with the help of quantum computers that can provide greater insights into climate change, and helping them if figuring out ways to tackle it.

- **Quantum Chemistry**: Even in the smallest of molecule, it is believed that the amount of quantum states is extremely vast, and therefore tough for traditional computing memory to process. Meanwhile quantum computers can focus on the existence of both 0 and 1 simultaneously which could provide a great amount of power to these machines to successfully map the molecules which potentially opens up opportunities in research related to pharmaceuticals. Some of the critical problems that could be solved via them are — improving the nitrogen-fixation process for creating ammonia-based fertilizer; creating a superconductor at room-temperature; removing carbon dioxide from the atmosphere for a better climate; and creation of solid-state batteries.

- **Financial Modelling:** Finding the right mix for fruitful investments for a finance industry is based on the risk associated, expected returns and other factors which are important for its survival in the market. To achieve that, the technique of 'Monte Carlo' simulations is continually being run on traditional computers, which consume an enormous amount of compute time. However, by applying quantum computing [12] in performing these huge and complex calculations, industries not only reduce the time to develop them but also improve upon the quality of the solutions.

- **Cybersecurity & Cryptography:** The online space recently has been quite vulnerable due to the increasing number of cyber-attacks occurring across the world. Although organizations are establishing essential security frameworks, the process has become unsettling and impractical for traditional computers. And, therefore, cybersecurity has continued to be an important concern round the world. We are becoming even more vulnerable to these threats with our increasing dependency on digitization. Quantum computers with the help of machine learning can help to develop several techniques to tackle these cybersecurity threats. Additionally, quantum computing [12] can help in creating encryption methods, also referred to as, quantum cryptography.

B. *Applications of Particle Swarm Optimization Algorithms (PSO) [10]*

- **Symbolic regression**, which may be a sort of multivariate analysis that searches the space of mathematical expressions to seek out the model that most closely fits a given dataset, both in terms of accuracy and ease, may be a crucially important theoretical and practical problem.

- Floor planning is to design **the layout of equipment in a factory or components on a computer chip** to reduce manufacturing time.

- **Weapon target assignment** problem is to seek out an optimal assignment of a group of weapons of varied types to a group of targets so as to maximize the expected damage done to the opponent.

- **Supply chain management** is the systemic and strategic coordination of the business functions and the tactics across them within a specific organization and across businesses within the availability chain, for the needs of improving the performance of the individual companies in the long run and therefore the supply chain as an entire.

- **Nurse scheduling problem** is to seek out an optimal path to assign nurses to shifts, typically with a group of hard constraints during which all valid solutions must follow and a set of sentimental constraints which define the relative quality of valid solutions.

- The mathematical study of waiting lines is named **queuing theory**. In queuing theory, a model is constructed so that queue lengths and waiting times can be predicted to make business decisions about the resources needed.

C. *Applications of Quantum Particle Swarm Optimization (QPSO) [11] and (FQPSO)*

- **Use of Quantum mechanics principles** instead of using Newtonian laws for governing particle motion unlike normal implementations of the Particle Swarm Optimization Algorithm (also known as the classical PSO implementations) [6].

- Quantum particle swarm optimization [8] can be used in order **to tune antennas more accurately**. It outperforms traditional PSO implementations and also has a much faster convergence rate than classical PSO.

- The QPSO algorithm can be used for predicting / designing a more efficient, **equivalent circuit** for a given Dielectric Resonator Antenna (DRA) circuit [16]. This can further make it easier to derive important circuit properties such as the **Quality factor** of the circuit (Q-factor).

- Both QPSO and the FQPSO algorithms have very few parameters which need to be tuned for optimization. This can be easily done by either checking accuracy at different states or values (i.e. trial and error) or by simple linear variation in the parameters. This also makes it easier to avoid errors and hence makes the algorithm even more accurate. Classical PSO generally involves 4 parameters: c_1 , c_2 , w , V_{max} . On the other hand, general implementations of the QPSO algorithms involve only one parameter which can be tuned easily for satisfactory convergence to optimal solution. This has the obvious advantage of less computational cost as tuning 4 parameters takes much more cycles than tuning a single parameter.

- The algorithm (QPSO) can also be used to find a set of infinitesimal dipoles that produce the same near as that produced by a circular dielectric resonator antenna or DRA [16].

- **The Dielectric Resonator Antenna** or DRA[16] is often used in microwaves. They generally consist of a block of ceramic material which acts as the dielectric. This block is mounted on a metal plate. Radiation is introduced inside the ceramic material by using a Transmitter. This radiation bounces around inside the ceramic block giving rise to standing radio waves. The walls of the ceramic block are preferentially transparent to radio-waves and hence let them escape into space giving rise to a radio signal.

- These antennas can have lower losses and be more efficient than metal antennas for high microwave and millimetre wave frequencies. This is where **QPSO can be used in order to optimise frequency values [15]** for finding the most optimum frequency or a range of optimum frequencies.

- The application of Fractional Calculus [5] to the field of Quantum Particle Swarm Optimization [8] results in the FQPSO technique. This new algorithm combines the speed of Quantum Particle Swarm Optimization [8] as well as the long-term memory effects and non-locality of Fractional Calculus [5] leading to faster convergence of particles while avoiding local optimums. It therefore significantly enhances the global optimum finding ability of the algorithm. Hence it has been experimentally verified that Fractional Order Quantum Particle Swarm Optimization (FQPSO) outperforms the traditional integer-based Quantum Particle Swarm Optimization [8] in most of the cases.

VII. SUMMARY AND CONCLUSION

The world of Quantum mechanics might be a new addition to the realm of science, but it can be said beyond doubt that this new field has given rise to some powerful concepts, extendable to various areas of modern life, one of them being complex engineering optimization problems. The Quantum Particle Swarm Optimization Algorithm [8] is cross between the classical principles of Particle Swarm Optimization (PSO) [6] and the probability equations which govern the quantum world.

We further investigated yet another algorithm: The Fractional Order Quantum Particle Swarm Optimization, which happens to be a cross between its predecessor (the QPSO) and the realm of Fractional Calculus. Together they exhibit a display the potential advantages which fractional order systems are able to extend to different areas of modern life. By replacing the integer-based approach to QPSO with a fractional index we have thus been able to show the superiority of FQPSO algorithm over the QPSO algorithm over various natural, optimization problem. Another variant of this algorithm also known as the Fractional order Darwinian Particle Swarm Optimization [27], although not based on quantum mechanical principles, can help us see how, extending Fractional Calculus [5] to evolutionary algorithms [29] can lead to a significant increase in accuracy in evolutionary algorithms (use of fractional order system in natural simulation). The reader can feel free to refer to the references section for further reading and to keep exploring this vast intersection of Physics, Maths and Biology.

REFERENCES

[1]. Young's Double-Slit Exp , Retrieved from : https://openphysicsjournal.com/VOLUME/3/PAGE/122/FULLTEXT/
[2]. Grunewald-Letnikov derivative , Retrieved from : https://www.researchgate.net/publication/220481880_A_Study_of_the_Grunwald-Letnikov_Definition_for_Minimizing_the_Effects_of_Random_Noise_on_Fractional_Order_Differential_Equations
[3]. Schrodinger Equation , Retrieved from : https://www.researchgate.net/publication/262007560_A_Review_of_Schrodinger_Equation_Classical_Wave_Equation
[4]. Klein-Gordon Equation , Retrieved from : https://www.degruyter.com/view/journals/zna/71/9/article-p817.xml?language=en
[5]. Fractional Calculus , Retrieved from : https://www.researchgate.net/publication/289991482_Special_issue_on_Fractional_calculus_and_applications
[6]. PSO (Particle Swarm Optimization) , Retrieved from : https://www.spiedigitallibrary.org/conference-proceedings-of-spie/10322/1/Research-on-particle-swarm-optimization-algorithm-based-on-optimal-movement/10.1117/12.2265257.full
[7]. Local attractor based PSO , Retrieved from : https://www.researchgate.net/publication/275066977_Particle_Swarm_Optimization_Based_on_Local_Attractors_of_Ordinary_Differential_Equation_System
[8]. QPSO (Quantum Particle Swarm Optimization) , Retrieved from : https://www.hindawi.com/journals/mpe/2019/1805198/
[9]. FQPSO (Fractional Quantum Particle Swarm Optimization) , Retrieved form : https://www.researchgate.net/publication/333912195_Fractional-order_quantum_particle_swarm_optimization https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0218285
[10]. Applications of Particle Swarm Optimization , Retrieved from : https://www.hindawi.com/journals/mpe/2019/8164609/
[11]. Applications of QPSO , Retrieved form : http://ieeexplore.ieee.org/document/6852313
[12]. Quantum Computing , Retrieved from : https://ieeexplore.ieee.org/abstract/document/8991254
[13]. Applying Qubits In Real Life , Retrieved from : https://www.academia.edu/44373054/Atoms_Communication_Quantum_Computing
[14]. Trapped Ion Quantum Computing , Retrieved from : https://www.academia.edu/44372853/Trapped_Ions_to_Quantum_Computing
[15]. Application in Antenna Frequency Optimization , Retrieved from : https://www.researchgate.net/publication/3018797_Quantum_Particle_Swarm_Optimization_for_Electromagnetics https://core.ac.uk/download/pdf/44172152.pdf
[16]. DRA (Dielectric Resonator Antenna) , Retrieved from : https://www.tandfonline.com/doi/abs/10.1080/02522667.2020.1822043
[17]. Quantum Computing Application , Retrieved from :https://analyticsindiamag.com/top-applications-of-quantum-computing-everyone-should-know-about/
[18]. Improving Quantum Computers , Retrieved from : https://phys.org/news/2019-04-quantum.html
[19]. Google's Quantum Supremacy , Retrieved from : https://www.bbc.com/news/science-environment-50154993
[20]. Google's Sycamore Chip , Retrieved from : https://www.scientificamerican.com/article/hands-on-with-googles-quantum-computer/#:~:text=Google's%20quantum%20computing%20chip%2C%20dubbed,some%209.700199254740992%20quadrillion%20bit%20strings

[21]. Superposition and Entanglement , Retrieved from https://www.afr.com/technology/quantum-computing-101-what-s-superposition-entanglement-and-a-qubit-20191218-p53l2j https://www.researchgate.net/publication/2196987_Superposition_and_Entanglement_Pillars_of_Quantum_Information_Processing
[22]. Quantum Computing , Retrieved from : https://www.technologyreview.com/2019/01/29/66141/what-is-quantum-computing/ https://uwaterloo.ca/institute-for-quantum-computing/quantum-computing-101 https://www.sciencedaily.com/terms/quantum_computer.htm https://www.wired.com/story/quantum-computing-here-but-not-really/ https://arxiv.org/abs/0708.0261 https://iopscience.iop.org/article/10.1088/0034-4885/61/2/002/meta
[23]. Quantum Algorithms , Retrieved from : https://www.quantum-inspire.com/kbase/what-is-a-quantum-algorithm/
[24]. Why Quantum Computing ? , Retrieved from : https://www.ncbi.nlm.nih.gov/books/NBK538701/
[25]. Quantum Development Kit , Retrieved from : https://docs.microsoft.com/en-us/quantum/overview/overview
[26]. Swarm Intelligence , Retrieved from : http://www.swarmintelligence.org/tutorials.php#:~:text=Particle%20swarm%20optimization%20(PSO)%20is,as%20Genetic%20Algorithms%20(GA).
[27]. Particle Swarm Algorithms , Retrieved from : http://www.iitg.ac.in/rkbc/CE602/CE602/Particle%20Swarm%20Algorithms.pdf
[28]. Darwinian Particle Swarm Optimization , Retrieved from : https://www.researchgate.net/publication/220888367_Darwinian_Particle_Swarm_Optimization
[29]. Evolutionary Algorithms , Retrieved from : https://www.researchgate.net/publication/261842296_Evolutionary_Algorithms