A Study of Caristi's Fixed Point Theorem on Normed Space

Md. Abdul Mannan*, Halima Akter, Nazmun Nahar
Department of Mathematics, Uttara University, Dhaka, Bangladesh
Samiran Mondal
Department of Mathematics
Jashore University of Science and Technology, Jashore, Bangladesh
* Corresponding author:
Md. Abdul Mannan, Department of Mathematics, Uttara University, Dhaka, Bangladesh

Abstract:- This paper aims at treating a study of Caristi's fixed point theorem for mapping results that introduced in the setting of normed space. The classical Caristi's fixed point theorem is a generalization of this work.

***Keywords**:- Norm Space, Complete Norm Space, Operator, Transfinite induction, Zorn's lemma and Banach Fixed Point Theorem.*

I. INTRODUCTION AND PRELIMINARIES

It is conventional to this work motivated by some recent work on Caristi's fixed point theorem for mappings defined on metric spaces with a partial order or a graph. One of the most important theorems is the Caristi fixed point theorem and it's related to a complete normed space with the Banach fixed point theorem. Here, Banach contraction principle and Caristi's fixed point theorem [3] is important for nonlinear analysis. It's a modification of the ε-variational principle of Ekeland [1, 2]. Then the different directions of the Caristi's fixed point theorem have been investigated by several authors; see, for example, [4–17] and references therein. Finally, we study of Caristi's fixed point theorem (Theorem-3) were given as mention above.

In this paper, we will discuss a proof of Caristi's fixed point theorem using mapping results which is introduced in the setting of normed spaces.

Normed Spaces 1.1: A normed on X is a real function $\|\bullet\|: X \to R$ defined on X such that for any $x,\ y \in X$ and for all $\lambda \in K$.

i. $\|x\| \geq 0$

ii. $\|x\| = 0 \ if \ \ and \ \ only \ \ if \ \ x = 0.$

iii. $\|\lambda x\| = |\lambda| \|x\|$

iv. $\|x+y\| \leq \|x\| + \|y\| \quad (Triangle \ \ inequality)$

A norm on X defines a metric d on X which is given by $d(x,\ y) = \|x-y\|; \quad x,\ y \in X$ and is called the metric induced by the norm.

The normed space is denoted by $(X,\ \|\bullet\|)$ or simply by X

II. A STUDY OF CARISTI'S FIXED POINT THEOREM ON NORMED SPACE

Here, we present a study of Caristi's fixed point theorem for mapping results which is introduced in setting of normed spaces such as.

Theorem 2 .1 (Caristi [3]). Suppose that $(X,\ \|,\ \|)$ is a complete normed spaces and let $T: X \to X$ be a mapping such that $\|a - Ta\| \leq \phi(a) - \phi(Ta) \ \forall \ a \in X \ . \ . \ . \quad . \ . \ . \ . \ . \ . \ (i)$

Where $\phi: X \to [0,\ \infty)$ is a lower semi-continuous mapping. Then T has at least fixed point.

Proof: We consider that $Ta \neq a \quad \forall a \in X$.

A map $N(a): X \to 2^{x}$ (Power of X) by

$N(a) = \{b \in X : \|a-b\| \le \phi(a) - \phi(b)\}$.

From equation (i) we know that $Ta \in N(a)$ and hence $N(a) \neq \phi \quad \forall \quad a \in X$.

We observe that for each $b \in N(a)$,

Here $\phi(b) \le \phi(a)$ and $N(b) \subseteq N(a)$.

Let $b \in N(a)$ be given.

We have $\|a-b\| \le \phi(a) - \phi(b)$.

So we have $\phi(b) \le \phi(a)$.Since $N(b) \neq \phi$

Let $t \in N(b)$.Then $t \neq b$ and $\|b-t\| \le \phi(b) - \phi(t)$

It follows that $\phi(t) \le \phi(b) \le \phi(a)$ and hence $\|a-t\| \le \|a-b\| + \|b-t\| \le \phi(a) - \phi(t)$

Also we have $t \neq a$ .Indeed if $t = a$. Then $\phi(t) = \phi(a)$ .Since

$\|a-b\| \le \phi(a) - \phi(b) \le \phi(a) - \phi(t) = \phi(a) - \phi(a) = 0$

$\Rightarrow \|a-b\| = 0$

$\Rightarrow a - b = 0$

$\Rightarrow a = b$

This would implies $b = t$, a contradiction.

Hence $t \in N(a)$.Therefore we prove $N(b) \subseteq N(a)$

We construct a sequence $< a_n >$.

The induction for any point $a_1 \in X$

Let $a_n \in X$.

Then for $a_{n+1} \in N(a)$ such that $N(a_{n+1}) \le \inf_{t \in N(a_n)} \phi(t) + \frac{1}{n}, \quad n \in \mathrm{N}.$ ....(ii)

For any $n \in \mathrm{N}$, since $a_{n+1} \in N(a)$, we have

$$\|a_n - a_{n+1}\| \le \phi(a_n) - \phi(a_{n+1}) \qquad ....(iii)$$

So, $\phi(a_{n+1}) \le \phi(a_n), \quad \forall \quad n \in \mathrm{N}$.

If ϕ is bounded below, then

$$\alpha = \lim_{n \to \infty} \phi(a_n) = \inf_{n \in \mathrm{N}} \phi(a_n) \text{ exist .} \qquad ....(iv)$$

From (iii) and (iv) then we get

$$\|a_n - a_{n+1}\| \le \sum_{j=n}^{m-1} \|a_j - a_{j+1}\| \le \phi(a_n) - \alpha \qquad [\because m > n \ \ and \ \ m, n \in N]$$

Since $\lim_{n \to \infty} \phi(a_n) = \alpha$ we get, $\lim_{n \to \infty} \sup\{ \ \|a_n - a_m\|\} = 0 \qquad \forall \ m > n$.

Therefore $< a_n >$ is a Cauchy sequence in X .

Then $\exists \ v \in X \ \ s.t. \ a_n \to v$ as $n \to \infty$.

i. e ϕ is a lower semi-continuous.

From (iv) then we get

$$\phi(v) \leq \liminf_{n\to\infty} \phi(a_n) = \inf_{n\in\mathrm{N}} \phi(a_n) \leq \phi(a_j) \quad \forall \quad j \in \mathrm{N} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(v)$$

Then we show that $\bigcap_{n=1}^{\infty} \mathrm{N}(a_n) = \{v\}$

For $m > n$ with $m, n \in \mathrm{N}$

From (iii) and (iv) then we get

$$\|a_n - a_m\| \leq \sum_{j=1}^{m-1} \|a_j - a_{j+1}\| \leq \phi(a_n) - \phi(v) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(vi)$$

Since $a_m \to v$ as $m \to \infty$ the inequality (vi) implies

$$\|a_n - v\| \leq \phi(a_n) - \phi(v) \quad \forall n \in \mathrm{N} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(vii)$$

By (vii) we know $v \in \bigcap_{n=1}^{\infty} N(a_n)$

Hence $\bigcap_{n=1}^{\infty} N(a_n) \neq \phi$ and $N(v) \subseteq \bigcap_{n=1}^{\infty} N(a_n)$

From (ii) then we have

$$\|a_n - w\| \leq \phi(a_n) - \phi(w) \leq \phi(a_n) - \inf_{t\in N(a_n)} \phi(t) \leq \phi(a_n) - \phi(a_{n+1}) + \frac{1}{n}$$

$$\Rightarrow \|a_n - w\| \leq \phi(a_n) - \phi(a_{n+1}) + \frac{1}{n} \quad \forall n \in \mathrm{N}$$

Hence, $\lim_{n\to\infty} \|a_n - w\| = 0 \Rightarrow a_n - w = 0 \Rightarrow a_n = w$.

Then, $w = v$.

Therefore, $\bigcap_{n=1}^{\infty} \mathrm{N}(a_n) = \{v\}$

Since $N(v) \neq \phi$ and $n(v) \subseteq \bigcap_{n=1}^{\infty} N(a_n) = \{v\}$. We obtain $N(v) = \{v\}$

On the contrary, from (i) we know $Tv = v$

Hence T has a fixed point v in X. Q. E. D

Remark 2 .2.

a) Although the function ϕ is lowering semi-continuous, it does not deduce that N (a) is a closed subset of X.

b) A study of Caristi's fixed point theorem contains assigning norm in X.

III. CONCLUSION

Our aim is to discuss a study of Caristi's fixed point theorem on normed space. We hope that this work will be useful for functional analysis related to normed spaces and fixed point theory.

ACKNOWLEDGEMENT

I would like to thank my respectable teacher Prof. Dr. Moqbul Hossain for guidance throughout the research process.

Author contributions:

Authors have made equal contributions for articles.

REFERENCES

[1]. Ekeland, I: On the variational principle. J. Math. Anal. Appl. 47(2), 324-353 (1974)

[2]. Ekeland, I: Nonconvex minimization problems. Bull. Am. Math. Soc. 1, 443-474 (1979

[3]. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc., 215 (1976), 241-251. http://dx.doi.org/10.1090/s0002-9947-1976-0394329-4

[4]. H. Brézis, F.E. Browder, A general principle on ordered sets in nonlinear functional analysis, Adv. Math., 21 (1976), no. 3, 355-364. http://dx.doi.org/10.1016/s0001-8708(76)80004-7

[5]. F. E. Browder, On a theorem of Caristi and Kirk, in Proceedings of the Seminar on Fixed Point Theory and Its Applications, June 1975, 23-27,Academic Press, New York, 1976.

[6]. W.-S. Du, On Caristi type maps and generalized distances with applications, Abstract and Applied Analysis, **2013** (2013), Article ID 407219,1-8. http://dx.doi.org/10.1155/2013/407219

[7]. W.-S. Du, On Caristi-type mappings without lower semi-continuity assumptions, Journal of Fixed Point Theory and Applications, 17 (2015), no. 4, 733-752. http://dx.doi.org/10.1007/s11784-015-0253-0

[8]. W.-S. Du, A simple proof of Caristi's fixed point theorem without using Zorn's lemma and transnite induction, submitted.

[9]. K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990. http://dx.doi.org/10.1017/cbo9780511526152

[10]. D.H. Hyers, G. Isac, T.M. Rassias, Topics in Nonlinear Analysis and Applications, World Scientific, Singapore, 1997. http://dx.doi.org/10.1142/9789812830432

[11]. J. R. Jachymski, Caristi's fixed point theorem and selections of set-valued contractions, J. Math. Anal. Appl., **227** (1998), 55-67.http://dx.doi.org/10.1006/jmaa.1998.6074

[12]. M.A. Khamsi, W.A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory, Pure and Applied Mathematics, Wiley-Interscience, New York, 2001. http://dx.doi.org/10.1002/9781118033074

[13]. W.A. Kirk, Caristi's fixed point theorem and metric convexity, Colloq. Math., **36** (1976), no. 1, 81-86.

[14]. W.A. Kirk, N. Shahzad, Fixed Point Theory in Distance Spaces, Springer, Cham, 2014. http://dx.doi.org/10.1007/978-3-319-10927-5

[15]. L. Pasicki, A short proof of the Caristi theorem, Ann. Soc. Math. Polon. Series I: Comm. Math., **22** (1978), 427-428.

[16]. T. Suzuki, Generalized Caristi's fixed point theorems by Bae and others, J. Math. Anal. Appl., **302** (2005), 502-508. http://dx.doi.org/10.1016/j.jmaa.2004.08.019

[17]. W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, Japan, 2000.