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A Statistical Solution to Markov Matrix Chains

Dr. Ismail Abbas, PhD Senior Lecturer at MTC, Cairo University

Abstract:- In part 1, we propose a statistical technique to the solution of stationary eigenvectors of Markov chains that is more efficient and more precise than the classical algebraic method. However, it only fails when the Markov matrix is not invertible, which is also the case for the classical solution. In part 2, we propose an important principle valid for B-Matrix chains: [For a positive symmetric physical matrix, the sum of their eigenvalues powers is equal to the eigenvalue of their sum of the series of powers of the matrix]. This principle is validated numerically by the derivation of an important equation for the sum of the series of algebraic powers, namely, the series of infinite power $[(1 + Mx) / (1 + M)]^N$ is equal to (1 + Mx) / (M-Mx), where ∀x is an element of the interval [0, 1 [and M is a positive integer.

I. INTRODUCTION

The Markov transition matrix A and the transition matrix B [1, 2, 3] operate on the same principle of the recurrence relation and are either related or different from each other in many aspects, we divide so this article in two parts, one for each.

PART 1
Markov transition matrix

The Markov transition matrix A (nxn) is defined by two assumptions, [4]
i-All its entries $a_{i,j}$ are elements of the closed interval [0,1]
ii- The sum of the entries for all rows / or all columns is equal to 1.

However, it seems right and important to add a third condition, rarely mentioned before, which is,
iii-The matrix A should be invertible or non-singular.

In the case where the Markov transition matrix is not invertible, the steady-state solution of the Markov chain could diverge or converge towards erroneous results.

The current classical algebraic procedure and the new statistical techniques proposed fail to obtain the stationary eigenvectors at equilibrium (x1, x2, ..., xn) of the homogeneous Markov equation Ax = 0 when A is not invertible.

In the classical system of resolution of Markov matrix chains Ax = 0, by a linear system of n algebraic equations, it is then necessary to apply an additional normalization condition on the eigenvector, namely the sum of its elements:
x1 + x2 + ... + xn = 1 in order to specify the numerical values of the eigenvector solution itself.

To be precise and objective, we compare here the classical and statistical methods in a simple illustrative example where A is invertible, (Shobha Deepthi V [5])
{A =
.0 .8 .2
.6 0 .4
.8 .2 0

The solution on Quora Q/A for the steady state vector v takes place in 3 consecutive steps like,
v: The vector of stable probabilities also known as steady-state probabilities is a vector that satisfies,
vP = v
v - steady state vector
P - Transition matrix (A)

Step 1: vP = v => vP -v = 0 => v (P-I) = 0
v: [x y z]
I - Identity matrix
1 0 0
0 1 0
0 0 1

Step 2: Say that the vector v is [x y z], then we also know that x + y + z = 1
From Step 2, we get 3 equations,
-x + 0.6y + 0.8 z = 0
-0.8x -y + 0.2 z = 0
0.2 x + 0.4 y - z = 0

where x, y z are interdependent but not specified.

Step 3
We apply the complementary condition x + y + z = 1
therefore solve the above equations.

Shobha obtains x = 0.403, y = 0.36 and z = 0.22. }
Here we notice that the elements of the solution vector add 0.983 instead of 1.000 as supposed to be.
In addition, the solution method is long and tedius one.
In the following theory, we show how the proposed statistical method treats the problem faster and more accurately.

II. THEORY

We propose a direct statistical method to find the steady-state eigenvector of the matrix A which is the subject of this article. the proposed solution emerges from examining the analogy of the Markov chain solution to IC problems in closed systems and the B-Matrix chain used in the statistical solution of the energy density diffusion problem IC -BC, namely,
$U^N (x, t) = D^N (b + S) + B^{\wedge}N . S (x,0)$. . . . (1). . . . [1,2,3]

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where b is the BC vector arranged in the proper order and S (x,0) represents IC.
The Transfer matrix D^N is given by,
D^N = B + B^2 +B^3 + ...B^N . . . . (2)

However, it is obvious by analogy with Eq. 1 that the matrix A ^ N is the statistical solution of the Markov chains. For N large enough, A ^ N = A * is an idle matrix (called idempotent matrix in mathematical language) satisfying the equality A*=A * ^2 = A *^3 . . .etc

A ^ N quickly converges to the stationary state solution A *. For the case described above by Shobha we obtain in just 16 time steps the following,
A*= A^16
0.4035 0 .3685 0 .2280
0.4035 0.3685 0.2280
0.4035 0.3685 0.2280

A * is an ergodic matrix with all the columns having equal elements and all the rows represent a unique solution of stationary eigenvector whose eigenvalue = 1..

Here, the elements of the Eigenvector columns add up to 1.000 as supposed, showing that the precision is much higher than the solution given by Shobha Deepthi V [5] explained above.

III. III.APPLICATIONS AND NUMERICAL RESULTS

We first discuss the case when the Markov transition matrix is a singular one which is rarely discussed in the previous literature.

Let us take a simple example of Stackoverflow Q / A on the singular Markov transition matrix A (4X4), cited here as follows, [6]
{ I have the following snippet to calculate the steady state of a transition matrix (4X4):

```
import numpy as np
import scipy.linalg as la
if __name__ == "__main__":
   P = np.array([[0.5, 0.2 , 0.3, 0],
              [0.5, 0 , 0.1 , 0.4],
              [0.6, 0.1, 0, 0.3],
              [0.5, 0.2, 0.3, 0]])
   # Sanity check:
   assert np.sum(P, axis=1).all() == 1.0
   print la.eig(P,left=True)[1]
```

and it prints:

```
[[ -8.78275813e-01 -7.07106781e-01 -5.00000000e-01 1.47441956e-01]
 [ -2.51874610e-01 -1.58270385e-16 -5.00000000e-01 -2.94883912e-01]
 [ -3.50434239e-01 -2.60486675e-16 5.00000000e-01 -5.89767825e-01]
 [ -2.05880116e-01 7.07106781e-01 5.00000000e-01 7.37209781e-01]]
```

If I understand correctly the first column of this is indeed the steady state. It does not make sense to me for the probability of being in a state to be negative. What have I missed?}

Which is clearly wrong.
If we use the statistical solution technique it would resulte in a steady state transition matrix :
.5208 .1494 .2078 .1221
.5208 .1494 .2078 .1221
.5208 .1494 .2078 .1221
.5208 .1494 .2078 .1221
Which is also wrong.

Although the sum of the elements of the column of eigenvectors (.5208 .1494 .2078 .1221) is unity as presupposed, the solution itself is also false!

Both methods fail to resolve Markov chains when the Markov matrix itself is singular.

We can now go further to test the correctness and precision of the Statistical Solution Method in a more complicated application: Markov matrix A (10X10).

Let A be arbitrary chosen as,
A=
1-{{0.3, 0.1, 0.1, 0.0, 0.2, 0.4, 0.5, 0.2, 0.3, 0.1},

2- {0.1, 0.15, 0.1, 0.2, 0.1, 0, 0.1, 0, 0.1, 0.4}

3= {0, 0.1, 0.5, 0.1, 0, 0.1, 0, 0.1, 0, 0.1}

4-, {0.05, 0.2, 0.1, 0.05, 0.04, 0, 0.1, 0, 0.1, 0},

5- {0.05, 0.05, 0, 0.1, 0.06, 0.07, 0, 0.1, 0.2, 0.1},

6- {0.2, 0.2, 0.04, 0.05, 0.1, 0.03, 0.1, 0, 0.0, 0}

7-{0, 0.02, 0, 0.2, 0.3, 0.1, 0.05, 0.1, 0.1, 0.04},

8-{0.15, 0.08, 0.06, 0, 0.1, 0, 0, 0.05, 0.1, 0.06},

9-{0, 0.1, 0.05, 0.1, 0, 0.1, 0.05, 0.05, 0.09, 0.2},

10-{0.15, 0, 0.05, 0.2, 0.1, 0.2, 0.1, 0.4, 0.01, 0}})

For A * = A ^ 12, the convergence is carried out quickly and we obtain,
A*=
1-({{0.2209, 0.2209, 0.2209, 0.2209, 0.2209, 0.2209, 0.2209, 0.2209, 0.2209, 0.2209},
2-{0.1305, 0.1305, 0.1305, 0.1305, 0.1305, 0.1305, 0.1305, 0.1305, 0.1305, 0.1305},
3-{0.09560, 0.09562, 0.09571, 0.09562, 0.09559, 0.09561, 0.09559, 0.09562, 0.09559, 0.09562},
4-{0.06652, 0.06653, 0.06654, 0.06653, 0.06652, 0.06653, 0.06652, 0.06653, 0.06652, 0.06653},
5-{0.06756, 0.06756, 0.06755, 0.06756, 0.06756, 0.06756, 0.06756, 0.06756, 0.06756, 0.06756},

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6-{0.09380, 0.09380, 0.09378, 0.09380, 0.09380, 0.09380, 0.09380, 0.09380, 0.09380, 0.09380},
7- {0.06786, 0.06786, 0.06785, 0.06786, 0.06786, 0.06786, 0.06786, 0.06786, 0.06786, 0.06786},
8-{0.07357, 0.07357, 0.07356, 0.07357, 0.07357, 0.07357, 0.07357, 0.07357, 0.07357, 0.07357},
9-{0.06997, 0.06997, 0.06997, 0.06997, 0.06997, 0.06997, 0.06997, 0.06997, 0.06997, 0.06997},
10-{0.1137, 0.1137, 0.1136, 0.1137, 0.1137, 0.1137, 0.1137, 0.1137, 0.1137, 0.1137}})

Obviously, the ergodic Eigenvector columns are the same and their elements are,

{1-0.2209 ,, 2-0.1305, 3-0.09560, 4-0.06652, 5-0.06756, 6-0.09380, 7- 0.06786 ,, 8-0.07357,9-0.06997,10-0.1137} T
They sum up to 0. 99998, which is a striking precision.!

PART 2

B-Matrix Chains

The chains of matrix B are different from those of Markov. [1,2,3] They can be considered as a statistical physical modification of the mathematical Markov chains.While the Markov matrix chains are a transition matrix in time and concern the evolution of the IC of a closed mathematical system without boundary conditions or source / sink term, the matrix B is a transition matrix in 4D space- time for the energy density field towards steady stat equilibrium.

Contrary to Markov chains, the B chains have a place for both BC and IC in 2D and 3D setups.
It should be mentioned that matrix B has an important and interesting property:
If the matrix B is a statistical physical matrix and the eigenvalue of B is ev, that of B ^ 2 is ev2, that of B ^ 3 is ev3 ... etc,
then, we have :[1,2,3]
ev2 = ev1 ^ 2 and ev3 = ev ^ 3.,. . .evN = ev ^ N. . . . (3)

while when N approaches infinity, evN approaches zero which is a necessary condition for the convergence of matrix D in Eq.2.

Another additional property is the following principle, [1]
[For a positive symmetric physical matrix, the sum of their eigenvalues powers is equal to the eigenvalue of their sum of the series of powers of the matrix].

In other words If,
D = B + B ^ 2 + B ^ 3 .... . + B ^ N . . . . . .(2)
Then,
evD = ev + ev ^ 2 + ev ^ 3 +. ... + ev ^ N . . . (4)

Therefore, using (2), (4) we can evaluate the sum of the finite or infinite algebraic series from N = 1 to N tends to infinity.
In fact, we used (2) and (4) in a previous article [3] to show that: the series of infinite powers [(1 + x) / 2] ^ N is equal to (1 + x) / (1-x), ∀x∈ [0,1 [

We show hereafter that the same procedure can be extended to many cases of power series.

For example, we can show that,
[(1 + 2 x) / 3] ^ N is equal to (1 + 2 x) / (2-2 x), ∀x∈ [0,1 [. . . . . [5]Consider the transition matrix B (8X8) as follows,[7]
1-RO (1-RO) / 6 (1-RO) / 6 0.0 (1-RO) / 6 0.0 0.0 0.0
2- (1-RO) / 6 RO 0.0 (1-RO) / 6 0.0 (1-RO) / 6 0.0 0.0
3- (1-RO) / 6 0.0 RO (1-RO) / 6 0.0 0.0 (1-RO) / 6 0.0
4- 0.0 (1-RO) / 6 (1-RO) / 6 RO 0.0 0.0 0.0 (1-RO) / 6
5- (1-RO) / 6 0.0 0.0 0.0 RO (1-RO) / 6 (1-RO) / 6 0.0
6- 0.0 (1-RO) / 6 0.0 0.0 (1-RO) / 6 RO 0.0 (1-RO) / 6
7 - 0.0 0.0 (1-RO) / 6 0.0 (1-RO) / 6 0.0 RO (1-RO) / 6
8- 0.0 0.0 0.0 (1-RO) / 6 0.0 (1-RO) / 6 (1-RO) / 6 RO

The main diagonal entry RO is element of [0,1] and is assumed to be constant over the entire diagonal. It is of particular importance.

For example, in the matrix above, for RO = 0, we have ev = 2/3, ev2 = 4/9, ev3 = 8/27 ,. . . evN = (2/3) ^ N.. .
And for RO = 1/2, we have ev = 3/4, ev2 = 9/16 ,. . . . . evN = (3/4) ^ N, satisfying Equation (3)
And the summation transfer matrix D = B + B ^ 2 +. . . . . + B ^ N can be proved equal to, [1]
D = (I-B) ^ - 1- I .... (6)

For N large enough.
The numerical calculation ,when repeated for different RO shows that the eigenvalue of D (evD) which is the infinite sum of the series B, is
evD = (1 + 2RO) / (2-2RO) . . . (7)
for all RO element of [0,1[

By going further for different matrices A of different size, we can validate the more general formula where RO is replaced by x: the series of infinite power [(1 + Mx) / (1 + M)] ^ N is equal to (1 + Mx) / M (1-x), where ∀x is an element of [0, 1 [and M is a positive integer.

IV. CONCLUSION

In part 1, it is shown that the proposed statistical solution for markov chains is fast, stable and precise. if the Markov matrix is not invertible, there is no steady state solution at equilibrium and the new method fails as well as the other classical methods based on the resolution of a homogeneous system of algebraic equations.

In part 2, the application of the hypothesis used in [1,2,3] leads to a new formula for series of infinite powers, namely the sum $[(1 + Mx) / (1 + M)]^N$ is equal to (1 + Mx) / (M-Mx), which numerically proves the correctness of the hypothesis itself.

It actually validates the proposed principle: [For positive symmetric physical power matrices : [For a positive symmetric physical matrix, the sum of their eigenvalues powers is equal to the eigenvalue of their sum of the series of powers of the matrix].

This principle facilitates the search for summation solutions of many series of infinite algebraic powers such that the sum of [(1 + Mx) / (1 + M)] ^ N is equal to (1 + Mx) / M (1-x), where ∀x is an element of [0, 1 [and M is a positive integer.

We therefore recommend the improvement achieved by the proposed statistical method which is promising in many areas.

N.B. All computations in this article have been produced with Author's double precision algorithm to ensure maximum precision, as followed by Ref. 8 for example.

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