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A Matrix Property for Comparative Assessment of subsets of Complex Matrices characterized by a given set of Global Mass and Global Alignment Factors

Debopam Ghosh

Abstract:- The present research article develops a Matrix Property that provides a comparative analysis and assessment of Matrices belonging to appropriately defined subsets of the Complex matrix space $M_{m\times n}(C)$, from a nonlinear perspective. The associated Mathematical framework is developed and the introduced concepts and terminologies Illustrated with appropriate Numerical Examples.

Keywords:- *Global Mass Factor of a Matrix, Global Alignment Factor of a Matrix, Effective Global Mass Factor of a Matrix, Total Mass associated with a Matrix, Component Mass portions associated with a Matrix.*

Notations

- $M_{m\times n}(C)$ denotes the Complex Matrix space of Matrices of order m by n
- $R(A)$ denotes the Global Mass Factor associated with the matrix $A_{m\times n}$
- $C(A)$ denotes the Global Alignment Factor associated with the matrix $A_{m\times n}$
- $R_0(A)$ denotes the Effective Global Mass Factor associated with the matrix $A_{m\times n}$
- $H(A)$ denotes the Fundamental Matrix associated with the matrix $A_{m\times n}$
- $\hat{m}(A)$ denotes the Total mass of the matrix $A_{m\times n}$
- $\hat{\lambda}_+(A), \hat{\lambda}_-(A)$ denote the Component mass portions associated with the matrix $A_{m\times n}$
- $|+\rangle = (\frac{1}{\sqrt{2}})\begin{bmatrix}1\\1\end{bmatrix}$ and $|-\rangle = (\frac{1}{\sqrt{2}})\begin{bmatrix}1\\-1\end{bmatrix}$
- $|c|$ denotes the modulus of the complex number c
- $\{|e_1\rangle, |e_2\rangle, ..., |e_m\rangle\}$ denotes the standard Orthonormal basis in C^m and $\{|f_1\rangle, |f_2\rangle, ..., |f_n\rangle\}$ denotes the standard Orthonormal basis in C^n
- $c^{\bullet}$ denotes the complex conjugate of the complex number c
- $|V\rangle = \begin{bmatrix}v_1\\v_2\\.\\.\\v_p\end{bmatrix}_{p\times 1}, \langle V| = [v_1^{\bullet} \quad v_2^{\bullet} \quad . \quad . \quad v_p^{\bullet}]_{1\times p}, B = \left[b_{ij}\right]_{p\times p}, \langle V|B|V\rangle = \sum_{i=1}^{p}\sum_{j=1}^{p} b_{ij} v_i^{\bullet} v_j$

I. INTRODUCTION

The present research article discusses a Matrix property defined on subsets of the Complex Matrix space $M_{m\times n}(C)$, which are characterized by the ordered pair of the Global Mass Factor and the Global Alignment Factor associated with the matrix, denoted as (R, C). This matrix property, denoted as Ω, provides a comparative analysis of the intrinsic structural aspects of the matrices belonging to the particular set, from the standpoint of the effect of the individual and overall modulus terms of the matrix elements, and overlap of the phase terms associated with Individual matrix elements, the phase terms determine the overall alignment, i.e. orientation relative to the $\{|+\rangle, |-\rangle\}$ basis of C^2, the individual modulus terms determine the Global mass factor and the Effective Global mass factor associated with the matrix, which in turn, plays role in determining the distribution of the total mass associated with the matrix into Component mass portions. However, Ω being a nonlinear transformation, the presented mathematical framework addresses an issue of Matrix analysis from a nonlinear perspective, which can reveal intricacies of Matrix arrays that may not be apparent under a strictly linear algebraic approach to Matrix array structural analysis.

II. MATHEMATICAL FRAMEWORK AND ASSOCIATED ANALYSIS

$A \in M_{m\times n}(C), A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} |e_i\rangle\langle f_j|$, we have the following:

$a_{ij} = r_{ij}c_{ij}$, Where $r_{ij} = |a_{ij}|$, therefore we have $r_{ij} \geq 0$, $c_{ij} \in C$, $|c_{ij}| = 1$, we consider the following convention that in the case of zero matrix elements of matrix A : $a_{ij} = 0 \Rightarrow r_{ij} = 0, c_{ij} = 1$

We define the Global Mass Factor ($R(A)$), the Effective Global Mass Factor ($R_0(A)$) and the Global Alignment Factor $C(A)$, associated with the Matrix $A_{m\times n}$:

$$R(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij} \quad , \quad R_0(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}(1-\exp(-r_{ij})) \ ,$$

$$C(A) = c_{11}c_{12}....c_{1n}c_{21}c_{22}....c_{2n}.........c_{m1}c_{m2}....c_{mn} = \prod_{i=1}^{m}\prod_{j=1}^{n} c_{ij}$$

Clearly $C(A) \in C, |C(A)| = 1, \forall A \in M_{m\times n}(C)$

We define the set $\bar{M}(r,c)$ as follows:

$$\bar{M}(r,c) \subset M_{m\times n}(C), \bar{M}(r,c) = \{A \in M_{m\times n}(C) \mid A \neq 0_{m\times n}, R(A) = r, C(A) = c\}$$

, where we have the condition: $r > 0, c \in C, |c| = 1$

We now define the following derived properties, based on the properties defined above, as following:

- $\lambda_{ij} = (\frac{r_{ij}}{r_0})(1-\exp(-r_{ij}))$, where $r_0 = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}(1-\exp(-r_{ij}))$, is the numerical realization of the Effective Global Mass factor $R_0(A)$

Therefore, $\lambda_{ij} \geq 0, \forall i = 1,2,..m; j = 1,2,....,n$ and $\sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij} = 1$

- $x = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}(1-\exp(-r_{ij}))$, where we have : $x(A) \in (0,1), \forall A \in \bar{M}(r,c)$
- $\mu_{ij} = (\frac{r_{ij}}{r})$, where $r = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}$, is the numerical realization of the Global Mass factor $R(A)$

Therefore, $\mu_{ij} \geq 0, \forall i = 1,2,..m; j = 1,2,....,n$ and $\sum_{i=1}^{m}\sum_{j=1}^{n} \mu_{ij} = 1$

- $y = \sum_{i=1}^{m}\sum_{j=1}^{n} \mu_{ij}(1-\exp(-r_{ij}))$, where we have: $y(A) \in (0,1), \forall A \in \bar{M}(r,c)$
- $Q(A) = \begin{bmatrix} (\frac{1}{2})(x^2+y^2) & xy \\ xy & (\frac{1}{2})(x^2+y^2) \end{bmatrix}, \forall A \in \bar{M}(r,c)$

, and we have the following associated Matrix :

$$H(A) = I_{2\times 2} + Q(A) = \begin{bmatrix} 1+(\frac{1}{2})(x^2+y^2) & xy \\ xy & 1+(\frac{1}{2})(x^2+y^2) \end{bmatrix}$$

$H(A)$ is Hermitian, Positive Definite, $\forall A \in \bar{M}(r,c)$

The Matrix $H(A)$ has the following Spectral decomposition:

$H(A) = \hat{\lambda}_+(A)|+\rangle\langle+| + \hat{\lambda}_-(A)|-\rangle\langle-|$, where the eigenvalues of $H(A)$, termed as the Component Mass portions of the Matrix $A_{m\times n}$, has the following expressions:

$\hat{\lambda}_+(A) = 1+(\frac{1}{2})(x+y)^2$ and $\hat{\lambda}_-(A) = 1+(\frac{1}{2})(x-y)^2$

We have: $1 \leq \hat{\lambda}_-(A) < \hat{\lambda}_+(A) < 3$

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The Total mass of the Matrix $A_{m\times n}$, denoted as $\hat{m}(A)$, is defined as the trace of the Matrix $H(A)$:

$$\hat{m}(A) = \hat{\lambda}_+(A) + \hat{\lambda}_-(A) = 2 + (x^2 + y^2)$$

We define the vector $|C_0\rangle = (1/\sqrt{2})\begin{bmatrix} c \\ c^{\bullet} \end{bmatrix}$, where $|C_0\rangle \in C^2, \langle C_0|C_0\rangle = 1$, c is the Numerical realization of the Global Alignment Factor $C(A)$

We finally define the Matrix property of Interest through the following Nonlinear Transformation Ω:

$\Omega : \bar{M}(r,c) \mapsto [1,3)$, such that: $\Omega(A) = \langle C_0|H(A)|C_0\rangle$

Numerical Examples

1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$, we have the following summary of the numerical values for the matrix properties considered as part of the presented mathematical framework:

- $r(A) = 2$, $c(A) = 1$, $r_0(A) = 2(1-\exp(-1)) = 1.2642$ .... (up to 4 decimal places)
- $x(A) = y(A) = (1-\exp(-1)) = 0.6321$ .....(up to 4 decimal places)
- $\hat{m}(A) = 2[1+(1-\exp(-1))^2] = 2.7992$ .....(up to 4 decimal places)
- $\hat{\lambda}_-(A) = 1$, $\hat{\lambda}_+(A) = 1+2(1-\exp(-1))^2 = 1.7992$ .....(up to 4 decimal places)
- $(\frac{\hat{\lambda}_+(A)}{\hat{m}(A)})\times 100 = 64.27$, $(\frac{\hat{\lambda}_-(A)}{\hat{m}(A)})\times 100 = 35.73$ .....(up to 2 decimal places)
- $|C_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$
- $\Omega(A) = \hat{\lambda}_+(A) = 1+2(1-\exp(-1))^2 = 1.7992$ .....(up to 4 decimal places)

2) $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$, we have the following summary of the numerical values for the matrix

properties under consideration:

- $r(B) = 2$, $c(B) = 1$, $r_0(B) = 2(1-\exp(-2)) = 1.7293$ .....(up to 4 decimal places)
- $x(B) = y(B) = (1-\exp(-2)) = 0.8647$ .....(up to 4 decimal places)
- $\hat{m}(B) = 2[1+(1-\exp(-2))^2] = 3.4953$ .....(up to 4 decimal places)
- $\hat{\lambda}_-(B) = 1$, $\hat{\lambda}_+(B) = 1+2(1-\exp(-2))^2 = 2.4953$ .....(up to 4 decimal places)
- $(\frac{\hat{\lambda}_+(B)}{\hat{m}(B)})\times 100 = 71.39$, $(\frac{\hat{\lambda}_-(B)}{\hat{m}(B)})\times 100 = 28.61$ .....(up to 2 decimal places)
- $|C_0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$
- $\Omega(B) = \hat{\lambda}_+(B) = 1+2(1-\exp(-2))^2 = 2.4953$ .....(up to 4 decimal places)

Therefore, $A, B \in \bar{M}(r=2, c=1), \quad \Omega(A) < \Omega(B)$

III. DISCUSSION AND CONCLUSION

The Matrix property Ω provides a comparative, quantitative assessment of the Structural intricacies of matrices belonging to any such set $\bar{M}(r,c)$, where $r > 0, |c| = 1$, In the Numerical example considered above, both the matrices belong to the same subset $\bar{M}(r=2, c=1)$ but they are observed to be associated with significantly different mass portion distributions owing to different distribution of and contributions from the constituent matrix elements. In subsequent studies, the Matrix property Ω will be analyzed in more depth to understand more clearly its scope and limitations towards its applicability in real life Theoretical/Computational problems.

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