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Series Solution of Euler-Bernoulli Beam Subjected to Concentrated Load Using Homotopy Perturbation Method (HPM)

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Abstract:- This research work investigates the series solution of Euler-Bernoulli beam subjected to concentrated loads by employing Homotopy Perturbation Method (HPM). The governing partial differential equation was transformed into ordinary differential equation via Galerkin's Decomposition Procedure (GDP), the resulting ordinary differential equation was solved semi-analytically by employing HPM. Graphical representation of various deflections of the beam with respect to varing parameters were presented.

***Keywords and Phrases**: Euler-Bernoulli beam, beam, Load, series solution, Concentrated Load, Galerkin's Decomposition Procedure (GDP), Homotopy Perturbation*

I. INTRODUCTION

A beam is defined as a structure having one of its dimensions much larger than the other two. The axis of the beam is defined along that longer dimension, and a cross-section normal to this axis is assumed to smoothly vary along the span or length of the beam. Civil engineering structures often consist of an assembly or grid of beams with cross-sections having shapes such as T's or I's. A large number of machine parts also are beam-like structures: lever arms, shafts, etc. Finally, several aeronautical structures such as wings and fuselages can also be treated as thin-walled beams.

The theory of solid mechanics of beams, more commonly referred to simply as "beam theory", plays a very important role in structural analysis because it provides the designer with a simple tool to analyse numerous structures. Although more sophisticated tools, such as the finite element method, are now widely available for the stress analysis of complex structures, beam models are often used at a pre-design stage because they provide valuable insight into the behaviour of structures. Such calculations are also quite useful when trying to validate purely computational solutions.

Several beam theories have been developed based on various assumptions, and lead to different levels of accuracy. One of the simplest and most useful of these theories was first described by Euler and Bernoulli and is commonly called Euler-Bernoulli beam theory. A fundamental assumption of this theory is that the cross-section of the beam is infinetly rigid in its own plane, i.e. no deformations occur in the plane of the cross-section. Consequently, the in-plane displacement field can be represented simply by two rigid body translations and one rigid body rotation. This fundamental assumption deals only with in-plane displacements of the cross-section. Two additional assumptions deal with the out-of-plane displacements of the section: during deformation, the cross-section is assumed to remain plane and normal to the deformed axis of the beam. The Homotopy perturbation method was used by (Ganji D.,2006,2009)[1-2],(Abu-Hilal,2003),(Kozien,2013). Talked extensively on Beam. While series solution to Euler-Bernoulli beam subjected to concentrated load was given by (Inglis,1934;Lu,2003;Usman, 2003;Mehri,Davar and Rahmani, Nguyen,2011)[3-14].

II. MATHEMATICAL FORMULATION

Taking into account the vibration of a Euler-Bernoulli beam of finite length (L). The partial differential equation for the vibration of the overall system is given as.

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial x^2}+K(x)w(x,t)=F(x,t)$$

Let $w(x,t)=y$ so that (1) can be re-written as below

$$EI\frac{\partial^4 y}{\partial x^4}+\rho A\frac{\partial^2 y}{\partial x^2}+K(x)y=F(x,t)$$

where

$E=$ Young's modulus, $I=$ Moment of inertia of the cross section
$\rho=$ Density of the mass, $A=$ Area of the cross section of the beam

$EI =$ Rigidity of the beam, $K =$ Winkler's Foundation
$F(x,t) =$ Forced response with the initial conditions

$$w(x,0) = 0 \quad and \quad \frac{\partial w(x,0)}{\partial t} = 0$$

which can be re-written as

$$y(x,t)\Big|_{t=0} \quad , \quad \frac{\partial y}{\partial t}\Big|_{t=0} = 0$$

Boundary Conditions are

$$w(0,t) = w(L,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(L,t)}{\partial x^2} = 0$$

which can be re-written as

$$y|_{x\to 0} = y|_{x\to L} = \frac{\partial^2 y}{\partial x^2}\Big|_{x\to 0} = \frac{\partial^2 y}{\partial x^2}\Big|_{x\to L} = 0$$

III. DISCRETIZATION OF THE GOVERNING EQUATION

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial x^2} + K(x)y = F(x,t)$$

The above differential equation (5) can be converted into ordinary differential equation. Using the Galerkin's decomposition procedure to separate the spatial and temporal parts of the lateral displacement functions.

$$y(x,t) = \phi(x)\cdot w(t)$$

For the given boundary conditions

$$\phi(x) = e^{-\beta x}$$

Satisfies the four boundary conditions

Therefore,

$$\frac{\partial \phi}{\partial x} = -\beta e^{-\beta x}, \frac{\partial^2 \phi}{\partial x^2} = \beta^2 e^{-\beta x}, \frac{\partial^3 \phi}{\partial x^3} = -\beta^3 e^{-\beta x}, \frac{\partial^4 \phi}{\partial x^4} = \beta^4 e^{-\beta x}$$

which implies from $y = \phi(x)\cdot w(t)$ that

$$\frac{\partial y}{\partial x} = w(t)\frac{\partial \phi}{\partial x}, \frac{\partial^2 y}{\partial x^2} = w(t)\frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^3 y}{\partial x^3} = w(t)\frac{\partial^3 \phi}{\partial x^3}, \frac{\partial^4 y}{\partial x^4} = w(t)\frac{\partial^4 \phi}{\partial x^4}$$

If we substitute equation (8) into (9), we have

$$\frac{\partial y}{\partial x} = -\beta e^{-\beta x}w(t), \frac{\partial^2 y}{\partial x^2} = \beta^2 e^{-\beta x}w(t), \frac{\partial^3 y}{\partial x^3} = -\beta^3 e^{-\beta x}w(t), \frac{\partial^4 y}{\partial x^4} = \beta^4 e^{-\beta x}w(t)$$

Also,

$$\frac{\partial y}{\partial t} = \phi(x)\frac{\partial w(t)}{\partial t} = e^{-\beta x}\frac{\partial w(t)}{\partial t} \quad and \quad \frac{\partial^2 y}{\partial t^2} = \phi(x)\frac{\partial^2 w(t)}{\partial t^2} = e^{-\beta x}\frac{\partial^2 w(t)}{\partial t^2}$$

Substituting the respective derivatives in (10) and (11) into (5) gives

$$EI\{\beta^4 e^{-\beta x} w(t)\} + \rho A\{e^{-\beta x}\frac{\partial^2 w}{\partial t^2}\} + K\{e^{-\beta x} w(t)\} = F(x,t)$$

which gives

$$EI\beta^4 w(t) + \rho A\frac{\partial^2 w}{\partial t^2} + Kw(t) = F(x,t)e^{\beta x}$$

which could be rearranged as

$$\rho A\frac{\partial^2 w}{\partial t^2} + (EI\beta^4 + K)w(t) = F(x,t)e^{\beta x}$$

For a sinusoidal forcing function of the form

$$F(x,t) = F_0 e^{-\beta x} Sinw_0 t$$

On substituting (14) into (13), one can arrive at

$$\rho A\frac{\partial^2 w}{\partial t^2} + (EI\beta^4 + K)w(t) = F_0 Sinw_0 t$$

which can be written as

$$\frac{\partial^2 w}{\partial t^2} + \left(\frac{EI\beta^4 + K}{\rho A}\right)w = \frac{F_0}{\rho A}Sinw_0 t$$

which can be further written as

$$\frac{\partial^2 w}{\partial t^2} + \lambda w = F_{00} Sinw_0 t$$

where

$$\lambda = \left(\frac{EI\beta^4 + K}{\rho A}\right) \quad , \quad F_{00} = \frac{F_0}{\rho A}$$

IV. SOLUTION SCHEME

Constructing an homotopy perturbation for (16) gives

$$H(w,p) = (1-p)\ddot{w} + p[\ddot{w} + \lambda w - F_{00} Sinw_0 t]$$

$$\Rightarrow H(w,p) = \ddot{w} + \lambda p w - pF_{00} Sinw_0 t = 0$$

where

$$\begin{aligned} w(t) &= w_0(t) + pw_1(t) + p^2 w_2(t) + \cdots \\ w &= w_0 + pw_1 + p^2 w_2 + \cdots \\ w &= \sum_{i=0}^{\infty} p^i w_i \end{aligned}$$

substituting (19) into (17) gives

$$H(w,p) = (\ddot{w}_0 + p\ddot{w}_1 + p^2\ddot{w}_2 + \cdots) + \lambda p(w_0 + pw_1 + p^2 w_2 + \cdots) - pF_{00} Sinw_0 t = 0$$

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$$H(w,p) = \frac{\partial^2 w}{\partial t^2} + \lambda p w - pF_{00} Sin(w_0 t) = 0$$
$$= \frac{\partial^2 w}{\partial t^2}(w_0 + pw_1 + p^2 w_2 + \cdots) + \lambda p(w_0 + pw_1 + p^2 w_2 + \cdots) - pF_{00} Sinw_0 t = 0$$
$$= \left(\frac{\partial^2 w_0}{\partial t^2} + p\frac{\partial^2 w_1}{\partial t^2} + p^2 \frac{\partial^2 w_2}{\partial t^2} + \cdots\right) + \lambda p(w_0 + pw_1 + p^2 w_2 + \cdots) - pF_{00} Sinw_0 t = 0$$

$$= \frac{\partial^2 w_0}{\partial t^2} + p\frac{\partial^2 w_1}{\partial t^2} + p\lambda w_0 - pF_{00} Sinw_0 t + p^2 \frac{\partial^2 w_2}{\partial t^2} + p^2 \lambda w_1 + p^3 \lambda w_2 + \cdots = 0$$

which gives

$$p^0: \frac{\partial^2 w_0}{\partial t^2} = \ddot{w}_0 = 0, \qquad w_0(0) = 0, \dot{w}_0(0) = 0$$
$$p^1: \frac{\partial^2 w_1}{\partial t^2} + \lambda w_0 - F_{00} Sinw_0 t, \qquad w_1(0) = 0, \dot{w}_1(0) = 0$$
$$p^1: \frac{\partial^2 w_1}{\partial t^2} + \lambda w_0 = F_{00} Sinw_0 t$$
$$: \ddot{w}_1 + \lambda w_0 = F_{00} Sinw_0 t$$
$$p^2: \frac{\partial^2 w_2}{\partial t^2} + \lambda w_1, \qquad w_2(0) = 0, \dot{w}_2(0) = 0$$
$$: \ddot{w}_2 + \lambda w_1 = 0$$
$$p^3: \ddot{w}_3 + \lambda w_2, \qquad w_3(0) = 0, \dot{w}_3(0) = 0$$
$$\vdots$$
$$p^n: \ddot{w}_n + \lambda w_{n-1}, \qquad w_n(0) = 0, \dot{w}_n(0) = 0$$

By the initial condition $\frac{\partial y}{\partial t}|_{t=0} = 0$, $w_0 = 0$

From Equation (21)

$\ddot{w}_0 = 0$

Hence , $\dot{w}_0 = 0$ and $\ddot{w}_0 = 0$

For Equation (22)

$\ddot{w}_1 + \lambda w_0 = F_{00} Sinw_0 t$
$\Rightarrow \ddot{w}_1 - F_{00} Sinw_0 t = 0$ since $\lambda w_0 = 0$ by the initial condition

$\Rightarrow \ddot{w}_1 = F_{00} Sinw_0 t$
$\Rightarrow w_1 = F_{00} \int\int Sinw_0 t \, dtdt = \frac{-F_{00}}{w^2} Sinw_0 t + C_1 t + C_2$

Therefore, we have $w_1 = \frac{-F_{00}}{w_0^2} Sinw_0 t + \frac{F_{00}}{w_0}$

where $C_1 = \frac{F_{00}}{w_0}$ and $C_2 = 0$

For Equation (23)

$$\ddot{w}_2 + \lambda w_1 = 0$$

which gives

$$\ddot{w}_2 + \lambda\left(\frac{-F_{00}}{w_0^2} Sinw_0 t + \frac{F_{00}}{w_0}\right) = 0$$

$$\Rightarrow \ddot{w}_2 = \lambda\left(\frac{F_{00}}{w_0^2} Sinw_0 t - \frac{F_{00}}{w_0}\right)$$

Applying boundary conditions

$$w_2 = \frac{1}{2}\frac{F_{00}\lambda\left(-\frac{1}{3}w_0 t^3 - \frac{2Sin(w_0 t)}{w_0^2}\right)}{w_0^2} + \frac{F_{00}\lambda t}{w_0^3}$$

Therefore considering Equation (24)

$$\ddot{w}_3 + \lambda w_2 = 0$$

$$\ddot{w}_3 + \lambda\left(\frac{1}{2}\frac{F_{00}\lambda\left(-\frac{1}{3}w_0 t^3 - \frac{2Sin(w_0 t)}{w_0^2}\right)}{w_0^2} + \frac{F_{00}\lambda t}{w_0^3}\right) = 0$$

Applying boundary conditions

$$w_3 = \frac{1}{24}\frac{F_{00}\lambda^2\left(\frac{1}{5}t^5 w_0^3 - 4w_0 t^3 - \frac{24Sin(w_0 t)}{w_0^2}\right)}{w_0^4} + \frac{F_{00}\lambda^2 t}{w_0^5}$$

Using Homotopy Perturbation Method definition

$$w(t) = w_0(t) + pw_1(t) + p^2 w_2(t) + p^3 w_3(t)\cdots$$

where,

$$w_0(t) = 0$$

$$w_1 = \frac{-F_{00}}{w_0^2} Sinw_0 t + \frac{F_{00}}{w_0}$$

$$w_2 = \frac{1}{2}\frac{F_{00}\lambda\left(-\frac{1}{3}w_0 t^3 - \frac{2Sin(w_0 t)}{w_0^2}\right)}{w_0^2} + \frac{F_{00}\lambda t}{w_0^3}$$

$$w_3 = \frac{1}{24}\frac{F_{00}\lambda^2\left(\frac{1}{5}t^5 w_0^3 - 4w_0 t^3 - \frac{24Sin(w_0 t)}{w_0^2}\right)}{w_0^4} + \frac{F_{00}\lambda^2 t}{w_0^5}$$

Therefore, we have

$$w(t) = \frac{-F_{00}}{w_0^2} Sinw_0 t + \frac{F_{00}}{w_0} + \frac{1}{2}\frac{F_{00}\lambda\left(-\frac{1}{3}w_0 t^3 - \frac{2Sin(w_0 t)}{w_0^2}\right)}{w_0^2} + \frac{F_{00}\lambda t}{w_0^3} + \frac{1}{24}\frac{F_{00}\lambda^2\left(\frac{1}{5}t^5 w_0^3 - 4w_0 t^3 - \frac{24Sin(w_0 t)}{w_0^2}\right)}{w_0^4} + \frac{F_{00}\lambda^2 t}{w_0^5}$$

Recall that from (6)

$$y(x,t) = \phi(x) \cdot w(t)$$

So, with (7)

$$y(x,t) = \frac{-F_{00}}{w_0^2} Sinw_0 t + \frac{F_{00}}{w_0} + \frac{1}{2}\frac{F_{00}\lambda\left(-\frac{1}{3}w_0 t^3 - \frac{2Sin(w_0 t)}{w_0^2}\right)}{w_0^2} + \frac{F_{00}\lambda t}{w_0^3} + \frac{1}{24}\frac{F_{00}\lambda^2\left(\frac{1}{5}t^5 w_0^3 - 4w_0 t^3 - \frac{24Sin(w_0 t)}{w_0^2}\right)}{w_0^4} + \frac{F_{00}\lambda^2 t}{w_0^5}$$

V. DISCUSSION OF RESULTS

We give a consequential observation from the above scheme used and discuss this consequential results through graphs while we vary values of different parameters. The vertical deflection $y(x,t)$ is plotted against time(t) and length(x) as will be seen in this section.

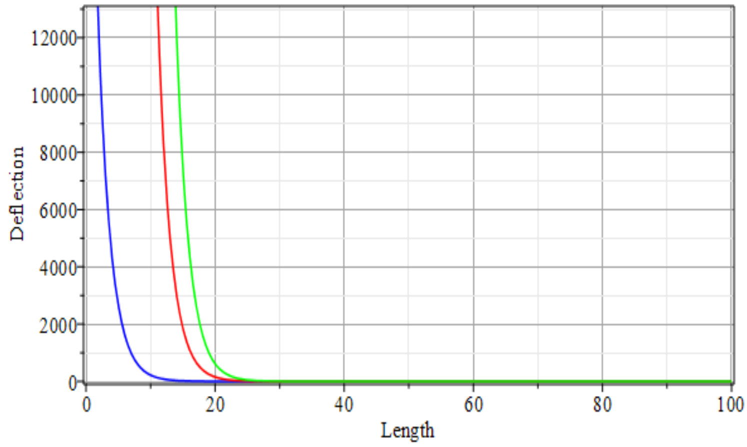

Figure 1: The graph of Deflection against length as λ is been varied while F_{00}, w_0 and β are kept constant

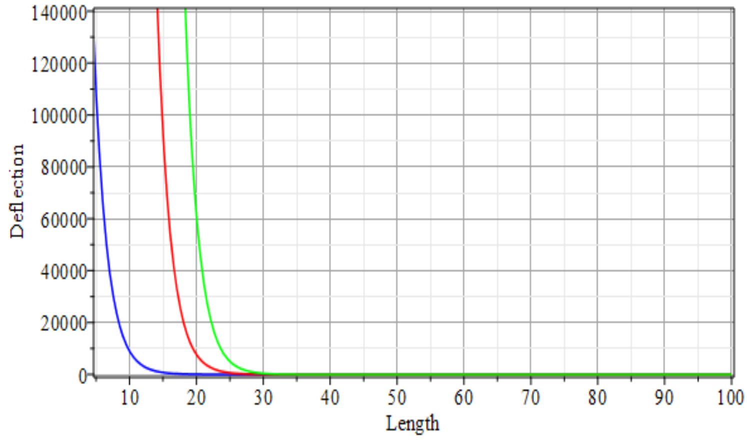

Figure 2: The graph of Deflection against length as λ and F_{00} is been varied simultaneously while w_0 and β are kept constant

From Figure 2 above as λ,(F_{00}) is been varied from lower value to higher value, we also see that the curve is an exponential curve. Which shows that varying both λ,(F_{00}) simultaneously has a tremendous effect which can be seen in figure 2. Where λ=F_{00}= 1000,5000, 10000 and w_0 = 0.4, β = 0.5.

In Figure 3 as w_0 is been varied from lower to higher value. We see that the graph almost gives a straight line which shows that the effect of w_0 on the system is very evident and cannot be ignored where w_0 = 10, 20, 30 and λ=F_{00}= 1000, , β = 0.015. We see that a small value of w_0 has a high defection value than when the value is higher for same length which can be seen in figure 3.

As λ is been varied from lower to higher value, we see the graph is a positive exponential curve. Unlike the graphs of Deflection against length which gives negative exponential curve. For a higher value of λ we see that deflection increases as time increases. tremendously other than small values of λ which is depicted in figure 4 and F_{00}= 100,
w_0= 0.4, β = 0:015 where the blue, red and green positive exponential curve corresponds

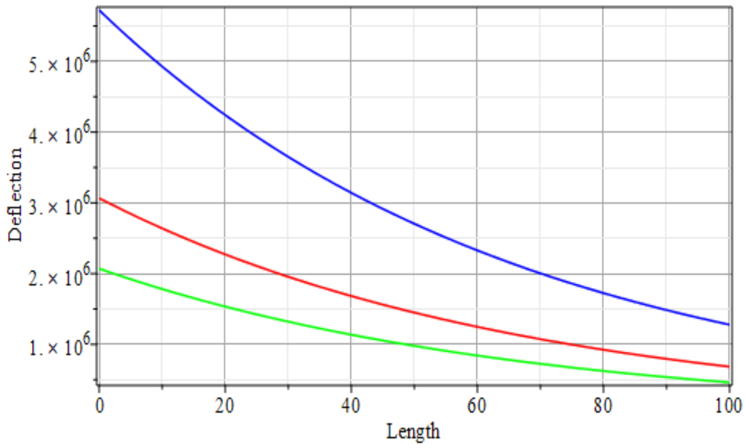

Figure 3: The graph of Deflection against length as w_0 is been varied while λ, F_{00} and β are kept constant

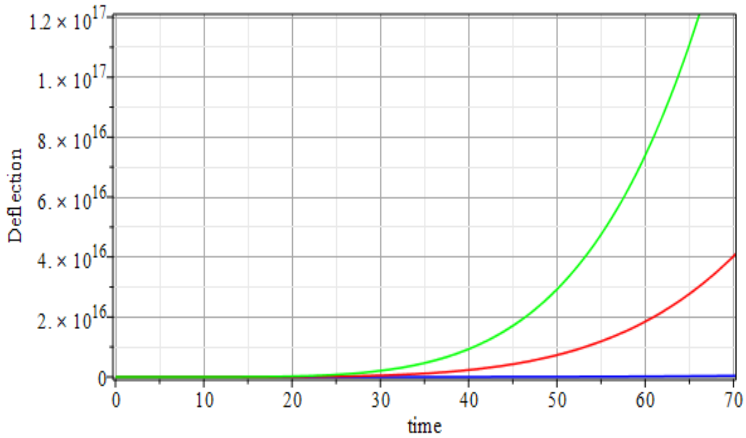

Figure 4: The graph of Deflection against length as λ is been varied while F_{00} , w_0 and β are kept constant to λ = 500, 5000, 10000 respectively.

As λ, F_{00} is been varied from lower to higher value, we see that the curve is a positive exponential curve, meaning that the varying effect of both parameter has a tremendous effect which can be seen in figure 5. where λ=F_{00}= 1000, 5000, 10000 and w_0= 0.4 and β = 0.015

As w_0 is been varied from lower to higher value, we see that the graph gives a positive exponential curve for each value corresponding to w_0 where w_0 = 10, 20, 30 and λ=F_{00}=

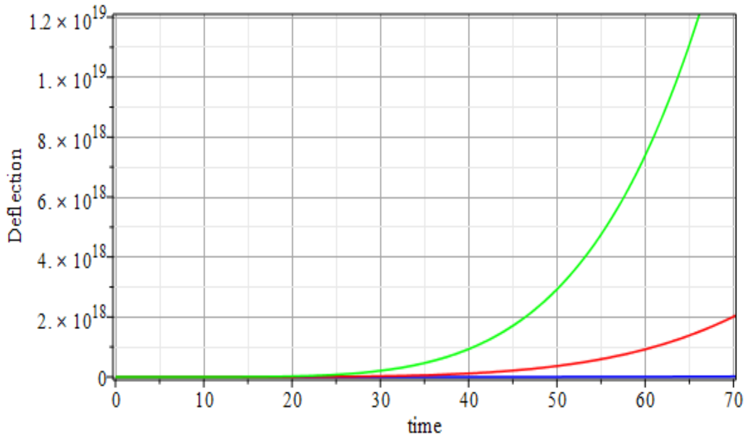

Figure 5: The graph of Deflection against time as λ and F_{00} is been varied simultaneously while w_0 and β are kept constant

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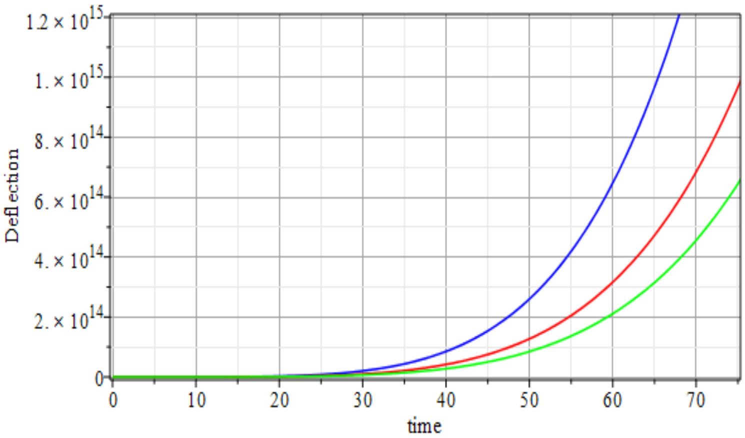

Figure 6: The graph of Deflection against time as w_0 is been varied while while λ, F_{00} and β are kept constant 10000 and β = 0:015 which is depicted in figure 6

VI. CONCLUSION

Homotopy Perturbation Method has been adopted as a semi-analytic method in providing a analytic solution to the governing equation in section 2. We can see that the system deflects with length and time as discussed in the discussion of results section when parameters are varied. So we say that a good result was reached by the Homotopy Perturbation Method used.

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