ISSN No:-2456-2165

A Generalized Matrix Multiplication Scheme based on the Concept of Embedding Dimension and Associated Spacer Matrices

Debopam Ghosh

Abstract:- The article presents a Generalized Matrix Multiplication scheme that allows Non-compatible matrices to be multiplied using Spacer matrices, which are unique for every possible pair of nonconforming dimensions. The article presents an approach to define positive integral powers of strictly rectangular complex matrices and thereby define subspaces of the Complex Matrix space using these matrix powers to constitute the pertinent spanning sets. The mathematical formalism is presented and Illustrated with numerical examples.

Keywords:- *Matrix Multiplication, Generalized Matrix Multiplication, Embedding Dimension, Spacer Matrix, Components of the Spacer Matrix*

Notations

- $M_{m\times n}(C)$ denotes the Complex Matrix space of order m by n
- The Matrices $A_{s\times m}$ and $B_{n\times t}$ are termed " Non-compatible " w.r.t. ordinary matrix multiplication AB when $m \neq n$ and for ordinary matrix multiplication BA when $s \neq t$
- N denotes the set of all Natural numbers
- r denotes the Embedding dimension
- $X_{m\times n}$ denotes the Spacer Matrix of order m by n
- $P_{m\times r}, Q_{r\times n}$ are the component Matrices associated with the Spacer Matrix $X_{m\times n}$
- $|m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{m\times 1}$, $\langle m| = \begin{bmatrix} 1 & 1 & . & . & 1 \end{bmatrix}_{1\times m}$
- $I_{s\times s}$ denotes the Identity Matrix of order 's'
- $\max(a,b)$ denotes the maximum of the two inputs a and b, $a,b \in N$
- $|a-b|$ denotes the absolute value of the difference of the two inputs a and b, $a,b \in N$
- The Generalized Matrix Multiplication operation is denoted by the symbol ' $\circ$ '
- A^T denotes the transpose of the Matrix A

I. INTRODUCTION

The present article addresses the issue of Matrix Multiplication pertaining to multiplication of Non-compatible matrices by introducing a Multiplication scheme that compensates for the non-compatibility by invoking the concept of an encapsulating higher dimension termed as "Embedding Dimension" and defining "Spacer matrices" associated with each such possible embedding dimension. The spacer matrices are so defined that they are independent of the matrix elements constituting the multiplication chain, and is unique for given pair of nonconforming dimensions, also, when there is compatibility these spacer matrices reduce to Identity matrices of appropriate orders, therefore, this generalized multiplication scheme is reduced to ordinary matrix multiplication scheme when there is compatibility among all the matrices that constitute the multiplication chain. The presented scheme allows the embedding dimension to be the same for a given pair of input dimensions regardless of their ordering in the ordered pair, and it also allows for defining the powers (positive and integral) of a strictly rectangular complex matrix $A_{m\times n}$, thereby allowing formation of Subspaces of the Complex matrix space $M_{m\times n}(C)$, using the unit matrix of order m by n , the matrix $A_{m\times n}$ and its defined powers as the corresponding spanning sets.

II. MATHEMATICAL FRAMEWORK AND ASSOCIATED ANALYSIS

$A \in M_{s\times m}(C)$, $B \in M_{n\times t}(C)$, we define the associated Embedding dimensions as follows:

$$u = \max(m,n) + |m-n| \quad , \quad v = \max(t,s) + |t-s|$$

The Spacer Matrix associated with dimension pair (m,n) is denoted by $X_{m\times n}$, which is defined as follows:

$X_{m\times n} = P_{m\times u} Q_{u\times n}$, where the space Matrix components have the following expressions:

- $P = I_{m\times m}$, when $m = n$

- $$P = \begin{bmatrix} I_{m\times m} & (\frac{1}{m})|m\rangle\langle u-m| \end{bmatrix}_{m\times u} = \left[\begin{array}{ccccc|ccccc} 1 & 0 & . & . & 0 & 1/m & 1/m & . & . & 1/m \\ 0 & 1 & . & . & 0 & 1/m & 1/m & . & . & 1/m \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 1/m & 1/m & . & . & 1/m \end{array}\right], \text{ when } m \neq n$$

- $Q = I_{n\times n}$, when $m = n$

- $$Q = \begin{bmatrix} I_{n\times n} \\ (\frac{1}{n})|u-n\rangle\langle n| \end{bmatrix}_{u\times n} = \left[\begin{array}{ccccc} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \\ \hline 1/n & 1/n & . & . & 1/n \\ 1/n & 1/n & . & . & 1/n \\ . & . & . & . & . \\ . & . & . & . & . \\ 1/n & 1/n & . & . & 1/n \end{array}\right] , \text{ when } m \neq n$$

The Spacer Matrix associated with dimension pair (t,s) is denoted by $Y_{t\times s}$, which is defined as follows:

$Y_{t\times s} = R_{t\times v} W_{v\times s}$, where the space Matrix components have the following expressions:

- $R = I_{t\times t}$, when $t = s$

- $$R = \begin{bmatrix} I_{t\times t} & (\frac{1}{t})|t\rangle\langle v-t| \end{bmatrix}_{t\times v} , \text{ when } t \neq s$$

- $W = I_{s\times s}$, when $t = s$
- $W = \begin{bmatrix} I_{s\times s} \\ (\frac{1}{s})|v-s\rangle\langle s| \end{bmatrix}_{v\times s}$, when $t \neq s$

Therefore, we define the Generalized Matrix Products $A \circ B$ and $B \circ A$ as follows:

$$A_{s\times m} \circ B_{n\times t} = A_{s\times m} X_{m\times n} B_{n\times t} = A_{s\times m} P_{m\times u} Q_{u\times n} B_{n\times t}$$

$$B_{n\times t} \circ A_{s\times m} = B_{n\times t} Y_{t\times s} A_{s\times m} = B_{n\times t} R_{t\times v} W_{v\times s} A_{s\times m}$$

<u>The Multiplication chain and the special case of chain of Identical Matrix units</u>

<u>General Scheme:</u>

$$(A_1)_{m_1\times n_1} \circ (A_2)_{m_2\times n_2} \circ .... \circ (A_{s-1})_{m_{s-1}\times n_{s-1}} \circ (A_s)_{m_s\times n_s} = A_1 (X_1)_{n_1\times m_2} A_2 (X_2)_{n_2\times m_3} .....A_{s-1} (X_{s-1})_{n_{s-1}\times m_s} A_s$$

Where we have the following:

$$(X_1)_{n_1\times m_2} = (P_1)_{n_1\times r_1} (Q_1)_{r_1\times m_2}, \quad r_1 = \max(n_1, m_2) + |n_1 - m_2|$$

$$(X_2)_{n_2\times m_3} = (P_2)_{n_2\times r_2} (Q_2)_{r_2\times m_3}, \quad r_2 = \max(n_2, m_3) + |n_2 - m_3|$$

Up to: $(X_{s-1})_{n_{s-1}\times m_s} = (P_{s-1})_{n_{s-1}\times r_{s-1}} (Q_{s-1})_{r_{s-1}\times m_s}, \quad r_{s-1} = \max(n_{s-1}, m_s) + |n_{s-1} - m_s|$

Numerical Examples:

$A_1 = (1)_{1\times 1}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$, $A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3}$, Then, we have the following:

$X_1 = \begin{pmatrix} \frac{3}{2} & \frac{3}{2} \end{pmatrix}_{1\times 2}$, $X_2 = \begin{pmatrix} \frac{7}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{7}{6} & \frac{2}{3} \end{pmatrix}_{2\times 3}$, Therefore we have:

$$(1)_{1\times 1} \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2} \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}_{1\times 3}$$

- $r = \max(m,n) + |m-n| = \max(n,m) + |n-m|, \forall m,n \in N$, Therefore, if the Spacer Matrix associated with dimension pair (m,n) be $X_{m\times n}$, $X_{m\times n} = P_{m\times r} Q_{r\times n}$

The Spacer Matrix associated with dimension pair (n,m) be $Y_{n\times m}$, $Y_{n\times m} = Q^T{}_{n\times r} P^T{}_{r\times m}$

Hence, we have: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} \circ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2} \circ (1)_{1\times 1} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}_{3\times 1}$

<u>Special case: Powers of a Strictly Rectangular Matrix</u>

$m_1 = m_2 = ... = m_s = m,\ n_1 = n_2 = ... = n_s = n,$ and we have $m \neq n$

Then, $r_1 = r_2 = ... = r_{s-1} = r,$ Where $r = \max(m,n) + |m-n|$

Therefore, the chain constituted by 's' such Matrix units $A_{m\times n}$, $A_{m\times n} \in M_{m\times n}(C)$, has the following Expression:

$A_{m\times n} \circ A_{m\times n} \circ .... \circ A_{m\times n} = A_{m\times n} X_{n\times m} A_{m\times n} .... A_{m\times n} X_{n\times m} A_{m\times n}$, Where $X_{n\times m} = P_{n\times r} Q_{r\times m}$

We can define powers of the strictly rectangular matrix $A_{m\times n}$ as follows:

$A_{m\times n} \circ A_{m\times n} = A_{m\times n} X_{n\times m} A_{m\times n}$, $A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} = A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n}$,

$A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} = A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n}$,

$A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} \circ A_{m\times n} = A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n} X_{n\times m} A_{m\times n}$

And so on.

Numerical Examples:

1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we have : $m = 2, n = 3, r = 4$

$$P_{3\times 4} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}, \quad Q_{4\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad X_{3\times 2} = \begin{bmatrix} \frac{7}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{7}{6} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$A \circ A = \begin{bmatrix} \frac{7}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{7}{6} & 0 \end{bmatrix}_{2\times 3}, \quad A \circ A \circ A = \begin{bmatrix} \frac{25}{18} & \frac{7}{18} & 0 \\ \frac{7}{18} & \frac{25}{18} & 0 \end{bmatrix}_{2\times 3},$$

$$A \circ A \circ A \circ A = \begin{bmatrix} 1.685185 & 0.685185 & 0 \\ 0.685185 & 1.685185 & 0 \end{bmatrix}_{2\times 3}$$ ... (Up to 6 decimal places)

$$A \circ A \circ A \circ A \circ A = \begin{bmatrix} 2.080247 & 1.080247 & 0 \\ 1.080247 & 2.080247 & 0 \end{bmatrix}_{2\times 3}$$ ... (Up to 6 decimal places)

2) $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, we have : $m = 2, n = 3, r = 4$

$$B \circ B = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix}, \quad B \circ B \circ B = \begin{bmatrix} 16 & 16 & 16 \\ 16 & 16 & 16 \end{bmatrix}, \quad B \circ B \circ B \circ B = \begin{bmatrix} 64 & 64 & 64 \\ 64 & 64 & 64 \end{bmatrix},$$

We have: $B \circ B \circ .... \circ B('s' times) = 4^{(s-1)} B$, where $s \geq 1,\ 4^0 = 1$

3) $C = \begin{bmatrix} -i & 0 & 0 \\ 0 & i & 0 \end{bmatrix}$, we have : $m = 2, n = 3, r = 4$

$$C \circ C = \begin{bmatrix} -\frac{7}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & -\frac{7}{6} & 0 \end{bmatrix}, \quad C \circ C \circ C = \begin{bmatrix} \frac{4i}{3} & 0 & 0 \\ 0 & -\frac{4i}{3} & 0 \end{bmatrix},$$

$$C \circ C \circ C \circ C = \begin{bmatrix} 1.555556 & -0.222222 & 0 \\ -0.222222 & 1.555556 & 0 \end{bmatrix}$$ .... (Up to 6 decimal places)

<u>The Fundamental Subspaces associated with a Non-zero, Strictly Rectangular Matrix</u>

$$A \in M_{m\times n}(C), \quad A \neq 0_{m\times n}, \quad m \neq n$$

The Fundamental subspace of $A_{m\times n}$, of degree '0' is denoted by $V(0)$:

$$V(0) = span(|m\rangle\langle n|)$$

The Fundamental subspace of $A_{m\times n}$, of degree '1' is denoted by $V(1)$:

$$V(1) = span(|m\rangle\langle n|, A_{m\times n})$$

The Fundamental subspace of $A_{m\times n}$, of degree '2' is denoted by $V(2)$:

$$V(2) = span(|m\rangle\langle n|, A_{m\times n}, A_{m\times n} \circ A_{m\times n}) = span(|m\rangle\langle n|, A_{m\times n}, A_{m\times n} X_{n\times m} A_{m\times n})$$

Under the considerations above, we have the following:

$V(3) = span(|m\rangle\langle n|, A_{m\times n}, A\circ A, A\circ A\circ A) = span(|m\rangle\langle n|, A_{m\times n}, A_{m\times n}X_{n\times m}A_{m\times n}, A_{m\times n}X_{n\times m}A_{m\times n}X_{n\times m}A_{m\times n})$ and so on.

We can see that: $V(s) \subseteq M_{m\times n}(C), \forall s \geq 0$, where $s \in \{0\} \cup N$,
$1 \leq \dim.of(V(s)) \leq m.n, \forall s \geq 0, s \in \{0\} \cup N$

Numerical Example:

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we have the following associated results:

$$V(0) = span(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}),\ \dim.(V(0)) = 1$$

$$V(1) = span(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}),\ \dim.(V(1)) = 2$$

$$V(2) = span(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{7}{6} & 0 \end{bmatrix}), \dim.(V(2)) = 3$$

$$V(3) = span(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} \frac{7}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{7}{6} & 0 \end{bmatrix}, \begin{bmatrix} \frac{25}{18} & \frac{7}{18} & 0 \\ \frac{7}{18} & \frac{25}{18} & 0 \end{bmatrix}), \dim.(V(3)) = 3$$

Table of 'm', 'n' and 's', for $m = 1, 2, 3, 4, 5$ **and** $n > m, \quad n \leq 10$

$$s = \max(m, n) + |m - n| = \max(n, m) + |n - m|$$

ISSN No:-2456-2165

m	n	s
1	2	3
	3	5
	4	7
	5	9
	6	11
	7	13
	8	15
	9	17
	10	19
2	3	4
	4	6
	5	8
	6	10
	7	12
	8	14
	9	16
	10	18
3	4	5
	5	7
	6	9
	7	11
	8	13
	9	15
	10	17
4	5	6
	6	8
	7	10
	8	12
	9	14
	10	16
5	6	7
	7	9
	8	11
	9	13
	10	15

Invariance under the Generalized Multiplication operation for Matrix powers in the case of nonconforming dimension pairs

$E_{m\times n} = (\frac{1}{r})|m\rangle\langle n|$, where $r = \max(m,n) + |m-n|,\ \ m \neq n$

Then, we have the following:

$$E_{m\times n} \circ E_{m\times n} = E_{m\times n} X_{n\times m} E_{m\times n} = E_{m\times n}$$

$$E_{m\times n} \circ E_{m\times n} \circ E_{m\times n} = E_{m\times n} X_{n\times m} E_{m\times n} X_{n\times m} E_{m\times n} = E_{m\times n}$$

In general, $E_{m\times n} \circ E_{m\times n} \circ \ldots E_{m\times n}('s'times) = E_{m\times n}, \forall s \geq 1, s \in N$

III. DISCUSSION AND CONCLUSION

The Generalized Multiplication scheme allows for Matrix Multiplication to be defined on any arbitrary set of Complex Matrices, and thereby extends the formalism of ordinary matrix multiplication, the spacer matrices are key components of this formalism; they compensate for any present non-compatibilities and is unique for any ordered pair of nonconforming dimensions and more importantly, independent of the matrix elements of the matrices that constitute the multiplication chain. When the ordering of the dimensions in the pair is interchanged, the resulting spacer matrix is just the transpose of the former. A particularly useful feature of this formalism is that it allows defining positive integral powers of a strictly rectangular complex matrix, The Unit matrix of order m by n is particularly interesting in this regard since all such powers of this matrix are multiple of itself(belonging to the same one dimensional matrix subspace associated with the unit matrix), thus, considering the unit matrix, the non-zero strictly rectangular matrix $A_{m\times n}$ and its integral powers, one can construct the largest possible Linearly Independent set(whose cardinality is bounded above by m.n, i.e. the dimension of the complex matrix space $M_{m\times n}(C)$, such a set forms a basis for a subspace of the complex matrix space $M_{m\times n}(C)$, which is determined completely by the matrices $|m\rangle\langle n|$, $A_{m\times n}$, $X_{n\times m}$ and the generalized multiplication operation '$\circ$', analysis of such subspaces can provide insights in theoretical/computational problems associated with such complex matrix spaces, which will be taken up in subsequent follow up studies.

REFERENCES

Books:

[1]. Anderson, T.W., *An Introduction to Multivariate Statistical Analysis*, 3rd Edition, Wiley-India Edition.

[2]. Arfken, George B., and Weber, Hans J., *Mathematical Methods for Physicists*, 6th Edition, Academic Press

[3]. Biswas, Suddhendu, *Textbook of Matrix Algebra*, 3rd Edition, PHI Learning Private Limited

[4]. Datta, B. N., *Numerical Linear Algebra and Applications*, SIAM

[5]. Ghosh, Debopam, *A Tryst with Matrices: The Matrix Shell Model Formalism, 24by7 Publishing, India.*

[6]. Graham, Alexander, *Kronecker Products & Matrix Calculus with Applications*, Dover Publications, Inc.

[7]. Halmos, P. R., *Finite Dimensional Vector Spaces*, Princeton University Press

[8]. Hassani, Sadri, *Mathematics Physics A Modern Introduction to its Foundations*, Springer

[9]. Hogben, Leslie, (Editor), *Handbook of Linear Algebra*, Chapman and Hall/CRC, Taylor and Francis Group

[10]. Johnson, Richard. A., and Wichern, Dean. W., *Applied Multivariate Statistical Analysis*, 6th Edition, Pearson International

[11]. Joshi, D. D., *Linear Estimation and Design of Experiments*, 2009 Reprint, New Age International Publishers

[12]. Meyer, Carl. D., *Matrix Analysis and Applied Linear Algebra*, SIAM

[13]. Rao, A. Ramachandra., and Bhimasankaram, P., *Linear Algebra*, 2nd Edition, Hindustan Book Agency

[14]. Rencher, Alvin. C., and Schaalje, G. Bruce., *Linear Models in Statistics*, John Wiley and Sons, Inc., Publication

[15]. Strang, Gilbert, *Linear Algebra and its Applications*, 4th Edition, Cengage Learning

[16]. Sundarapandian, V., *Numerical Linear Algebra*, PHI Learning Private Limited

Research Articles:

[1]. Brewer, J. W., *Kronecker Products and Matrix Calculus in System Theory*, IEEE Trans. on Circuits and Systems, 25, No.9, p 772-781 (1978)

[2]. Kecilioglu, Osman, and Gundogan, Halit, *GENERALIZED MATRIX MULTIPLICATION AND ITS SOME APPLICATION*, FACTA UNIVERSITATIS (NIS), SER.MATH.INFORM.

[3]. Vol.32, No.5 (2017), p 789-798

[4]. Neudecker, H., *Some Theorems on Matrix Differentiation with special reference to Kronecker Matrix Products*, J. Amer. Statist. Assoc., 64, p 953-963(1969)

[5]. Paris, Matteo G A, *The modern tools of Quantum Mechanics : A tutorial on quantum states, measurements and operations*, arXiv: 1110.6815v2 [quant-ph] (2012)

[6]. Roth, W. E., *On Direct Product Matrices*, Bull. Amer. Math. Soc., No. 40, p 461-468 (1944)

[7]. Slyusar, V.,I., *A FAMILY OF FACE PRODUCTS OF MATRICES AND ITS PROPERTIES*, Cybernetics and Systems Analysis, Vol.35, No.3, (1999)

[8]. Slyusar, V.,I., *GENERALIZED FACE-PRODUCTS OF MATRICES IN MODELS OF DIGITAL ANTENNA ARRAYS WITH NONIDENTICAL CHANNELS*, Radioelectronics and Communication Systems, Vol.46, No.10, p 9-17, (2003)