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The Reduced Description framework associated with the Matrix Shell Model Formalism

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Abstract:- The present research article formulates a Mathematical Framework that simplifies the computational requirements associated with the Matrix Shell Model Formalism [5],[25]. This is achieved by removal of the computations involving the Matrix Shell Interaction coefficients and setting them all to unity, additional simplification is achieved by introducing appropriate " Averaging" at the level of Pathways which results in a single, Hermitian matrix (The Reduced State-Interaction Matrix description) being associated with every constituent Matrix Shells. This ultimately results in a single, Hermitian matrix (The Reduced State-Interaction Matrix description of the Matrix Shell system) for the given numerical realization of the Fundamental matrix associated with the Matrix Shell System 'N'. Numerical Illustrations are drawn from U(2) subset of Complex Matrix space of order '2', The set of Analytical Expressions pertaining to the Overlap elements associated with N=2 Matrix Shell system in a U(2) Matrix numerical realization is also compiled and presented in this paper.

***Keywords**:- The Matrix Shell Model Formalism, State-Interaction matrices associated with Matrix Shell System Configurations, Reduced State-Interaction Matrix description of a Matrix Shell, Directional states associated with a Pathway, State interaction Matrix description of the Matrix Shell System 'N', Reduced State-Interaction Matrix Description of the Matrix Shell system 'N'*

Notations

- N_0 denotes the set of all natural numbers
- C denotes the set of all complex numbers
- $N = 2n$ denotes the Even type Matrix Shell systems, n=1,2,3,.... , $n \in N_0$
- $N = 2n-1$ denotes the Odd type Matrix Shell systems, n=1,2,3,.... , $n \in N_0$
- $A(\lambda, N)$ denotes the Matrix Shells associated with the Matrix Shell system 'N' , where $\lambda = 0,1,...(n-1)$
- $R(\lambda, N)$ denotes the Reduced State-Interaction Matrix Description of the Matrix Shell $A(\lambda, N)$
- $R(N)$ denotes the Reduced State-Interaction Matrix Description of the Matrix Shell System 'N'
- $A_{N\times N}$ denotes the Fundamental Matrix associated with the Matrix Shell system "N"
- $\bar{\theta}(N)$ denotes the Reduced Effective mass of the Matrix Shell System 'N'
- $\bar{w}(N)$ denotes the Reduced Weightage function associated with the Matrix Shell system 'N'
- $M_{s\times s}(C)$ denotes the Complex Matrix space of order 's'
- $Tr(X_{s\times s})$ denotes the trace of the matrix $X_{s\times s}$
- $U(2)$ denotes the set of all Unitary matrices of order 2
- $c^{\bullet}$ denotes the complex conjugate of the complex number c
- $|c|$ denotes the modulus of the complex number c
- $\{|e_1\rangle, |e_2\rangle, ..., |e_s\rangle\}$ denotes the standard Orthonormal basis in C^s
- $|W\rangle = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_q \end{bmatrix}_{q\times 1}, \langle V| = [v_1^{\bullet} \quad v_2^{\bullet} \quad . \quad . \quad v_p^{\bullet}]_{1\times p}, B = \left[b_{ij}\right]_{p\times q}, \langle V|B|W\rangle = \sum_{i=1}^{p}\sum_{j=1}^{q} b_{ij} v_i^{\bullet} w_j$

- Inertia($A_{s\times s}$) = $\{n_+, n_0, n_-\}$, $n_+ + n_0 + n_- = s$, $A_{s\times s}$ is Hermitian, $A_{s\times s} \in M_{s\times s}(C)$, n_+ = number of positive eigenvalues of the Matrix $A_{s\times s}$, n_0 = number of zero eigenvalues of the Matrix $A_{s\times s}$, n_- = number of negative eigenvalues of $A_{s\times s}$.

I. INTRODUCTION

The Matrix Shell Model formalism [5], [25], provides a visualization of a Square Complex Matrix as a Numerical structure composed of Interpenetrating Abstract mathematical entities termed as “Matrix Shells”, it presents a visual analogy to peels of an onion where the “Onion” is visualized as the Square Matrix with the feature of Directionality incorporated (The Matrix Shell System) and the “Onion peels” are visualized as constituent Matrix Shells. This formalism provides an alternate view to Numerical square matrices and matrix manipulations/computations.

In Matrix Shell Model, structural intricacies in visualization lead to defining a set of tuning coefficients, i.e. the Matrix Shell Interaction coefficients for each contributing Matrix shell, this result in drastic increase in number of computations associated with each numerically active Matrix Shells. Additionally, the concept of “Pathway Directionality” manifests as 64 possible Configurations (32 possible State-Interaction Matrix descriptions) for each numerically active Matrix Shell, it can be observed that this diversity feature in Matrix shell framework results in an extremely large number of computations being associated with the complete Shell system. In [25], a framework is proposed using the criteria of “Ordered Eigen spectral Proximity” to Individual, contributing, Matrix Shell Baseline Matrix Ordered eigenspectrum and that of the Matrix Shell Baseline Matrix associated with the Matrix Shell System ‘N’.

In this research article, attempt is made to reduce the number of computations involved by simplifying the Matrix Shell Model framework, this is achieved by assuming every contributing Matrix Shell is saturated, i.e. the Matrix Shell Interaction coefficients associated with each such Matrix Shell set to unity. Additional simplification in formalism is achieved by a proposed form of “Averaging” at the level of the Individual directional states, this ultimately leads to a single State-Interaction Matrix description (The Reduced State - Interaction Matrix description, which is Hermitian and belongs to the Matrix Space $M_{6\times 6}(C)$). Therefore, for a Matrix Shell System ‘N’ (N=2n type or N=2n-1 type) constituted by Matrix Shells $A(0,N), A(1,N),...., A(n-1,N)$ we have the respective Reduced State-Interaction Matrix Descriptions $R(0,N), R(1,N),..., R(n-1,N)$, which by linear combination forms the Reduced State- Interaction Matrix Description of the Matrix Shell System, denoted as $R(N)$.

The Numerical Illustrations presented in this research article are drawn from the U(2) Subset of the Matrix Space $M_{2\times 2}(C)$. An U(2) Matrix, in light of the Matrix Shell Model formalism becomes a N=2 Matrix Shell System (constituted by only one Matrix Shell A(0,2)). The Analytical Expressions for the overlap elements associated with N=2 Matrix Shell System in U(2) Matrix realization is presented in this paper. The Numerical examples discussed in this article include the Pauli matrices of order 2, The Hadamard Matrix of order 2 and Identity matrix of order 2 and some linear combinations of some these matrices. The computed matrix is subjected to determination of their rank, Inertia and the ordered eigenspectrum, the results obtained are discussed and the article concludes with observations on directions for further research on the research problem presented in this paper.

II. MATHEMATICAL FRAMEWORK AND RELATED ANALYSIS

- The Notations used in this article, unless otherwise stated and defined in the notation section above, follow that used in [5], [25]

The key features of the Reduced Description framework pertaining to the Matrix Shell Model formalism [5], [25]:

- The Interaction Framework is assumed to be “saturated”, i.e. all the Matrix Shell Interaction coefficients set to unity, for each contributing Matrix Shell associated with the Matrix Shell system.
- The framework of Effective mass and Weightage function of the Matrix Shell system ‘N’ is replaced by their simplified descriptions as “Reduced Effective mass” and “Reduced Weightage function”, respectively.
- The framework of Directional states associated with Individual pathways of a constituent Matrix shell is retained, and an appropriate “averaging” is performed at the level of each Individual pathways, this leads to removal of the possibility of different configurations (64 such configurations in the actual framework) and replacing them with a single, “Reduced State-Interaction Matrix” description for each constituent, contributing Matrix shells.

- The Reduced State-Interaction Matrix description of each constituent Matrix Shell is Hermitian and belongs to the Complex Matrix Space $M_{6\times 6}(C)$. Linear combination of the Reduced State-Interaction Matrices corresponding to the constituent Matrix shells forms the "Reduced State-Interaction Matrix" description of the Matrix Shell System 'N'.

The Analytical Expressions for the π co-efficients:

- $\pi_1 = \left[\frac{1}{2}(|\langle 1|3\rangle|^2) + \frac{1}{2}(|\langle 1|P|3\rangle|^2)\right]^{1/2}$, $\pi_2 = \left[\frac{1}{2}(|\langle 1|5\rangle|^2) + \frac{1}{2}(|\langle 1|P|5\rangle|^2)\right]^{1/2}$
- $\pi_3 = \left[\frac{1}{2}(|\langle 3|5\rangle|^2) + \frac{1}{2}(|\langle 3|P|5\rangle|^2)\right]^{1/2}$, $\pi_4 = \left[\frac{1}{2}(|\langle 2|4\rangle|^2) + \frac{1}{2}(|\langle 2|P|4\rangle|^2)\right]^{1/2}$
- $\pi_5 = \left[\frac{1}{2}(|\langle 2|6\rangle|^2) + \frac{1}{2}(|\langle 2|P|6\rangle|^2)\right]^{1/2}$, $\pi_6 = \left[\frac{1}{2}(|\langle 4|6\rangle|^2) + \frac{1}{2}(|\langle 4|P|6\rangle|^2)\right]^{1/2}$
- $\pi_7 = \left[\frac{1}{2}(|\langle 1|2\rangle|^2) + \frac{1}{2}(|\langle 1|P|2\rangle|^2)\right]^{1/2}$, $\pi_8 = \left[\frac{1}{2}(|\langle 1|4\rangle|^2) + \frac{1}{2}(|\langle 1|P|4\rangle|^2)\right]^{1/2}$
- $\pi_9 = \left[\frac{1}{2}(|\langle 1|6\rangle|^2) + \frac{1}{2}(|\langle 1|P|6\rangle|^2)\right]^{1/2}$, $\pi_{10} = \left[\frac{1}{2}(|\langle 3|2\rangle|^2) + \frac{1}{2}(|\langle 3|P|2\rangle|^2)\right]^{1/2}$
- $\pi_{11} = \left[\frac{1}{2}(|\langle 3|4\rangle|^2) + \frac{1}{2}(|\langle 3|P|4\rangle|^2)\right]^{1/2}$, $\pi_{12} = \left[\frac{1}{2}(|\langle 3|6\rangle|^2) + \frac{1}{2}(|\langle 3|P|6\rangle|^2)\right]^{1/2}$
- $\pi_{13} = \left[\frac{1}{2}(|\langle 5|2\rangle|^2) + \frac{1}{2}(|\langle 5|P|2\rangle|^2)\right]^{1/2}$, $\pi_{14} = \left[\frac{1}{2}(|\langle 5|4\rangle|^2) + \frac{1}{2}(|\langle 5|P|4\rangle|^2)\right]^{1/2}$
- $\pi_{15} = \left[\frac{1}{2}(|\langle 5|6\rangle|^2) + \frac{1}{2}(|\langle 5|P|6\rangle|^2)\right]^{1/2}$

The Analytical Expressions for the χ co-efficients:

- $\chi_1 = \frac{\langle 1|3\rangle}{|\langle 1|3\rangle|}$, $|\langle 1|3\rangle| = 0 \Rightarrow \chi_1 = 1$, $\chi_2 = \frac{\langle 1|5\rangle}{|\langle 1|5\rangle|}$, $|\langle 1|5\rangle| = 0 \Rightarrow \chi_2 = 1$
- $\chi_3 = \frac{\langle 3|5\rangle}{|\langle 3|5\rangle|}$, $|\langle 3|5\rangle| = 0 \Rightarrow \chi_3 = 1$, $\chi_4 = \frac{\langle 2|4\rangle}{|\langle 2|4\rangle|}$, $|\langle 2|4\rangle| = 0 \Rightarrow \chi_4 = 1$
- $\chi_5 = \frac{\langle 2|6\rangle}{|\langle 2|6\rangle|}$, $|\langle 2|6\rangle| = 0 \Rightarrow \chi_5 = 1$, $\chi_6 = \frac{\langle 4|6\rangle}{|\langle 4|6\rangle|}$, $|\langle 4|6\rangle| = 0 \Rightarrow \chi_6 = 1$
- $\chi_7 = \frac{\langle 1|2\rangle}{|\langle 1|2\rangle|}$, $|\langle 1|2\rangle| = 0 \Rightarrow \chi_7 = 1$, $\chi_8 = \frac{\langle 1|4\rangle}{|\langle 1|4\rangle|}$, $|\langle 1|4\rangle| = 0 \Rightarrow \chi_8 = 1$
- $\chi_9 = \frac{\langle 1|6\rangle}{|\langle 1|6\rangle|}$, $|\langle 1|6\rangle| = 0 \Rightarrow \chi_9 = 1$, $\chi_{10} = \frac{\langle 3|2\rangle}{|\langle 3|2\rangle|}$, $|\langle 3|2\rangle| = 0 \Rightarrow \chi_{10} = 1$
- $\chi_{11} = \frac{\langle 3|4\rangle}{|\langle 3|4\rangle|}$, $|\langle 3|4\rangle| = 0 \Rightarrow \chi_{10} = 1$, $\chi_{12} = \frac{\langle 3|6\rangle}{|\langle 3|6\rangle|}$, $|\langle 3|6\rangle| = 0 \Rightarrow \chi_{10} = 1$
- $\chi_{13} = \frac{\langle 5|2\rangle}{|\langle 5|2\rangle|}$, $|\langle 5|2\rangle| = 0 \Rightarrow \chi_{13} = 1$, $\chi_{14} = \frac{\langle 5|4\rangle}{|\langle 5|4\rangle|}$, $|\langle 5|4\rangle| = 0 \Rightarrow \chi_{14} = 1$

- $\chi_{15} = \dfrac{\langle 5|6\rangle}{|\langle 5|6\rangle|}$, $|\langle 5|6\rangle| = 0 \Rightarrow \chi_{15} = 1$

The Analytical Expressions for the ψ co-efficients:

- $\psi_1 = \dfrac{\langle 1|P|3\rangle}{|\langle 1|P|3\rangle|}$, $|\langle 1|P|3\rangle| = 0 \Rightarrow \psi_1 = 1$, $\psi_2 = \dfrac{\langle 1|P|5\rangle}{|\langle 1|P|5\rangle|}$, $|\langle 1|P|5\rangle| = 0 \Rightarrow \psi_2 = 1$
- $\psi_3 = \dfrac{\langle 3|P|5\rangle}{|\langle 3|P|5\rangle|}$, $|\langle 3|P|5\rangle| = 0 \Rightarrow \psi_3 = 1$, $\psi_4 = \dfrac{\langle 2|P|4\rangle}{|\langle 2|P|4\rangle|}$, $|\langle 2|P|4\rangle| = 0 \Rightarrow \psi_4 = 1$
- $\psi_5 = \dfrac{\langle 2|P|6\rangle}{|\langle 2|P|6\rangle|}$, $|\langle 2|P|6\rangle| = 0 \Rightarrow \psi_5 = 1$, $\psi_6 = \dfrac{\langle 4|P|6\rangle}{|\langle 4|P|6\rangle|}$, $|\langle 4|P|6\rangle| = 0 \Rightarrow \psi_6 = 1$
- $\psi_7 = \dfrac{\langle 1|P|2\rangle}{|\langle 1|P|2\rangle|}$, $|\langle 1|P|2\rangle| = 0 \Rightarrow \psi_7 = 1$, $\psi_8 = \dfrac{\langle 1|P|4\rangle}{|\langle 1|P|4\rangle|}$, $|\langle 1|P|4\rangle| = 0 \Rightarrow \psi_8 = 1$
- $\psi_9 = \dfrac{\langle 1|P|6\rangle}{|\langle 1|P|6\rangle|}$, $|\langle 1|P|6\rangle| = 0 \Rightarrow \psi_9 = 1$, $\psi_{10} = \dfrac{\langle 3|P|2\rangle}{|\langle 3|P|2\rangle|}$, $|\langle 3|P|2\rangle| = 0 \Rightarrow \psi_{10} = 1$
- $\psi_{11} = \dfrac{\langle 3|P|4\rangle}{|\langle 3|P|4\rangle|}$, $|\langle 3|P|4\rangle| = 0 \Rightarrow \psi_{11} = 1$, $\psi_{12} = \dfrac{\langle 3|P|6\rangle}{|\langle 3|P|6\rangle|}$, $|\langle 3|P|6\rangle| = 0 \Rightarrow \psi_{12} = 1$
- $\psi_{13} = \dfrac{\langle 5|P|2\rangle}{|\langle 5|P|2\rangle|}$, $|\langle 5|P|2\rangle| = 0 \Rightarrow \psi_{13} = 1$, $\psi_{14} = \dfrac{\langle 5|P|4\rangle}{|\langle 5|P|4\rangle|}$, $|\langle 5|P|4\rangle| = 0 \Rightarrow \psi_{14} = 1$
- $\psi_{15} = \dfrac{\langle 5|P|6\rangle}{|\langle 5|P|6\rangle|}$, $|\langle 5|P|6\rangle| = 0 \Rightarrow \psi_{15} = 1$

The Analytical Expressions for the derived co-efficients**:**

- $x_1 = \langle 1|1\rangle = \langle \bar{1}|\bar{1}\rangle$, $x_2 = \langle 3|3\rangle = \langle \bar{3}|\bar{3}\rangle$, $x_3 = \langle 5|5\rangle = \langle \bar{5}|\bar{5}\rangle$
- $x_4 = \langle 2|2\rangle = \langle \bar{2}|\bar{2}\rangle$, $x_5 = \langle 4|4\rangle = \langle \bar{4}|\bar{4}\rangle$, $x_6 = \langle 6|6\rangle = \langle \bar{6}|\bar{6}\rangle$
- $b_1 = \pi_1\chi_1\psi_1$, $b_2 = \pi_2\chi_2\psi_2$, $b_3 = \pi_3\chi_3\psi_3$, $b_4 = \pi_4\chi_4\psi_4$, $b_5 = \pi_5\chi_5\psi_5$, $b_6 = \pi_6\chi_6\psi_6$
- $v_1 = \pi_7\chi_7\psi_7$, $v_2 = \pi_8\chi_8\psi_8$, $v_3 = \pi_9\chi_9\psi_9$, $v_4 = \pi_{10}\chi_{10}\psi_{10}$, $v_5 = \pi_{11}\chi_{11}\psi_{11}$, $v_6 = \pi_{12}\chi_{12}\psi_{12}$
- $v_7 = \pi_{13}\chi_{13}\psi_{13}$, $v_8 = \pi_{14}\chi_{14}\psi_{14}$, $v_9 = \pi_{15}\chi_{15}\psi_{15}$

The Analytical Expression for the $R(\lambda, N)$ Matrix:

$$R(\lambda, N) = \begin{bmatrix} x_1 & b_1 & b_2 & v_1 & v_2 & v_3 \\ b_1^{\bullet} & x_2 & b_3 & v_4 & v_5 & v_6 \\ b_2^{\bullet} & b_3^{\bullet} & x_3 & v_7 & v_8 & v_9 \\ v_1^{\bullet} & v_4^{\bullet} & v_7^{\bullet} & x_4 & b_4 & b_5 \\ v_2^{\bullet} & v_5^{\bullet} & v_8^{\bullet} & b_4^{\bullet} & x_5 & b_6 \\ v_3^{\bullet} & v_6^{\bullet} & v_9^{\bullet} & b_5^{\bullet} & b_6^{\bullet} & x_6 \end{bmatrix}_{6\times 6}$$

$R(\lambda, N) \in M_{6\times 6}(C)$

The Analytical Expression for the $R(N)$ Matrix:

- $\bar{\theta}(N)=\sum_{i=1}^{N}\sum_{j=1}^{N}\left|a_{ij}\right|$, where $A_{N\times N}=\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\left|e_i\right\rangle\left\langle e_j\right|$, $[\bar{w}(N)]^0=1$
- $\bar{w}(N)=1-\exp(-\bar{\theta}(N))$
- $R(N)=\sum_{\lambda=0}^{(n-1)}[\bar{w}(N)]^{\lambda}R(\lambda,N)=R(0,N)+[1-\exp(-\bar{\theta}(N))]R(1,N)+....+[1-\exp(-\bar{\theta}(N))]^{(n-1)}R(n-1,N)$

Clearly, $R(N)$ is Hermitian, $R(N)\in M_{6\times6}(C)$

<u>The Analytical Expressions for the Overlap elements associated with a general U(2) Matrix</u>

$$U(2)=\{U(\gamma,\alpha,\beta)\in M_{2\times2}(C)\,|\,U(\gamma,\alpha,\beta)=e^{+i\gamma}\begin{bmatrix}\alpha & -\beta^{\bullet}\\ \beta & \alpha^{\bullet}\end{bmatrix},\gamma\in[0,2\pi),\alpha\in C,\beta\in C,|\alpha|^2+|\beta|^2=1\}$$

- $\langle 1|1\rangle=\langle\bar{1}|\bar{1}\rangle=2|\alpha|^2$, $\langle 3|3\rangle=\langle\bar{3}|\bar{3}\rangle=1$, $\langle 5|5\rangle=\langle\bar{5}|\bar{5}\rangle=1$
- $\langle 2|2\rangle=\langle\bar{2}|\bar{2}\rangle=2|\beta|^2$, $\langle 4|4\rangle=\langle\bar{4}|\bar{4}\rangle=1$, $\langle 6|6\rangle=\langle\bar{6}|\bar{6}\rangle=1$
- $\langle 1|3\rangle=|\alpha|^2-\alpha\beta^{\bullet}$, $\langle 1|5\rangle=|\alpha|^2+\alpha^{\bullet}\beta$, $\langle 3|5\rangle=0$
- $\langle 1|P|3\rangle=\alpha\alpha-\alpha^{\bullet}\beta^{\bullet}$, $\langle 1|P|5\rangle=\alpha^{\bullet}\alpha^{\bullet}+\alpha\beta$, $\langle 3|P|5\rangle=\alpha^{\bullet}\alpha^{\bullet}-\beta\beta$
- $\langle 2|4\rangle=|\beta|^2-\alpha\beta$, $\langle 2|6\rangle=|\beta|^2+\alpha^{\bullet}\beta^{\bullet}$, $\langle 4|6\rangle=0$
- $\langle 2|P|4\rangle=-\beta\beta+\alpha\beta^{\bullet}$, $\langle 2|P|6\rangle=-\beta^{\bullet}\beta^{\bullet}-\alpha^{\bullet}\beta$, $\langle 4|P|6\rangle=\alpha^{\bullet}\alpha^{\bullet}-\beta^{\bullet}\beta^{\bullet}$
- $\langle 1|2\rangle=\alpha\beta-\alpha^{\bullet}\beta^{\bullet}$, $\langle 1|4\rangle=|\alpha|^2+\alpha\beta$, $\langle 1|6\rangle=|\alpha|^2-\alpha^{\bullet}\beta^{\bullet}$
- $\langle 1|P|2\rangle=\alpha^{\bullet}\beta-\alpha\beta^{\bullet}$, $\langle 1|P|4\rangle=\alpha\alpha+\alpha^{\bullet}\beta$, $\langle 1|P|6\rangle=\alpha^{\bullet}\alpha^{\bullet}-\alpha\beta^{\bullet}$
- $\langle 3|2\rangle=-\beta\beta-\alpha^{\bullet}\beta^{\bullet}$, $\langle 3|4\rangle=|\alpha|^2-\beta\beta$, $\langle 3|6\rangle=-\alpha^{\bullet}\beta-\alpha^{\bullet}\beta^{\bullet}$
- $\langle 3|P|2\rangle=|\beta|^2+\alpha^{\bullet}\beta$, $\langle 3|P|4\rangle=\alpha^{\bullet}\beta-\alpha\beta$, $\langle 3|P|6\rangle=\alpha^{\bullet}\alpha^{\bullet}+|\beta|^2$
- $\langle 5|2\rangle=\alpha\beta-\beta^{\bullet}\beta^{\bullet}$, $\langle 5|4\rangle=\alpha\beta+\alpha\beta^{\bullet}$, $\langle 5|6\rangle=|\alpha|^2-\beta^{\bullet}\beta^{\bullet}$
- $\langle 5|P|2\rangle=|\beta|^2-\alpha\beta^{\bullet}$, $\langle 5|P|4\rangle=|\beta|^2+\alpha\alpha$, $\langle 5|P|6\rangle=\alpha^{\bullet}\beta^{\bullet}-\alpha\beta^{\bullet}$

Properties of the $R(N=2\,|\,A=U(\gamma,\alpha,\beta))$ Matrices:

- $U(\gamma,\alpha,\beta)\mapsto R(N=2\,|\,A=U(\gamma,\alpha,\beta))$, $R(N=2\,|\,A=U(\gamma,\alpha,\beta))\in M_{6\times6}(C)$, it is Hermitian and is independent of the 'γ' parameter.
- $Tr[R(N=2\,|\,A=U(\gamma,\alpha,\beta))]=6$, $\forall\gamma\in[0,2\pi),\alpha\in C,\beta\in C,|\alpha|^2+|\beta|^2=1$
- $R(N=2\,|\,A=U(\gamma,\alpha,\beta))=\Theta(\alpha,\beta)+\mathrm{K}(\alpha,\beta)$, Where $\Theta(\alpha,\beta)=diag(2|\alpha|^2,1,1,2|\beta|^2,1,1)$ and $\mathrm{K}(\alpha,\beta)$ is a zero-diagonal (i.e. main diagonal of all zeroes) matrix $\in M_{6\times6}(C)$

III. Numerical Examples

1) $U(\gamma,\alpha,\beta)=I_{2\times2}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$R(N=2\,|\,A=I_{2\times2})=\begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}_{6\times6}$$

rank [$R(N=2\,|\,A=I_{2\times2})$] = 5, Inertia[$R(N=2\,|\,A=I_{2\times2})$] = {5,1,0}
eigenspectrum [$R(N=2\,|\,A=I_{2\times2})$ = {4.637759, 0.483561, 0.292893, 0.292893, 0.292893, 0 }
... (upto 6 decimal places)

2) $U(\gamma,\alpha,\beta)=\Sigma_1=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times2}$

$$R(N=2\,|\,A=\Sigma_1)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}_{6\times6}$$

rank [$R(N=2\,|\,A=\Sigma_1)$] = 5 , Inertia[$R(N=2\,|\,A=\Sigma_1)$] = {5,1,0}
eigenspectrum [$R(N=2\,|\,A=\Sigma_1)$] = { 4.637759 , 0.483561, 0.292893, 0.292893, 0.292893,0 }
... (upto 6 decimal places)

3) $U(\gamma,\alpha,\beta)=\Sigma_2=\begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix}_{2\times2}$

$$R(N=2|A=\Sigma_2)=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 1 & -1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -1 & -1 & 2 & -1 & -1 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}_{6\times 6}$$

rank [$R(N=2|A=\Sigma_2)$] = 5 , Inertia[$R(N=2|A=\Sigma_2)$] = {4,1,1}
eigenspectrum [$R(N=2|A=\Sigma_2)$] = { 3.32097, 3.12132, 0.292893, 0.292893, 0, -1.028077 }
... (upto 6 decimal places)

4) $U(\gamma,\alpha,\beta)=\Sigma_3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2}$

$$R(N=2|A=\Sigma_3)=\begin{bmatrix} 2 & -1 & -1 & 0 & -1 & -1 \\ -1 & 1 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -1 & -\frac{1}{\sqrt{2}} & 1 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 & -\frac{1}{\sqrt{2}} \\ -1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}_{6\times 6}$$

rank [$R(N=2|A=\Sigma_3)$] = 5 , Inertia[$R(N=2|A=\Sigma_3)$] = {4,1,1}

eigenspectrum [$R(N=2|A=\Sigma_3)$] = { 3.32097, 3.12132, 0.292893, 0.292893, 0, -1.028077 }
... (upto 6 decimal places)

5) $U(\gamma,\alpha,\beta)=H_{2\times 2}=\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$R(N=2\,|\,A=H_{2\times2})=\begin{bmatrix}1&0&-1&0&0&-1\\0&1&0&1&1&0\\-1&0&1&0&0&-1\\0&1&0&1&1&0\\0&1&0&1&1&0\\-1&0&-1&0&0&1\end{bmatrix}_{6\times6}$$

rank [$R(N=2\,|\,A=H_{2\times2})$] = 4 , Inertia[$R(N=2\,|\,A=H_{2\times2})$] = {3,2,1}

eigenspectrum [$R(N=2\,|\,A=H_{2\times2})$] = { 3,2,2,0,0,-1 }

6) $$U(\gamma,\alpha,\beta)=\frac{1}{\sqrt{2}}(I_{2\times2}+i\Sigma_1)=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}\\\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}_{2\times2}$$

$$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(I_{2\times2}+i\Sigma_1))=\begin{bmatrix}1&\frac{i}{\sqrt{2}}&\frac{i}{\sqrt{2}}&-1&\frac{i}{\sqrt{2}}&\frac{i}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}&1&\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&1&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-1&-\frac{i}{\sqrt{2}}&-\frac{i}{\sqrt{2}}&1&-\frac{i}{\sqrt{2}}&-\frac{i}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}&1&\frac{1}{\sqrt{2}}\\-\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&\frac{i}{\sqrt{2}}&\frac{1}{\sqrt{2}}&1\end{bmatrix}_{6\times6}$$ rank

[$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(I_{2\times2}+i\Sigma_1))$] = 5 , Inertia[$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(I_{2\times2}+i\Sigma_1))$] = {5,1,0}

eigenspectrum [$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(I_{2\times2}+i\Sigma_1))$] = { 4.637759, 0.483561, 0.292893, 0.292893, 0.292893, 0 } .... (upto 6 decimal places)

7) $$U(\gamma,\alpha,\beta)=\frac{1}{\sqrt{2}}(\Sigma_2+\Sigma_3)=\begin{bmatrix}\frac{1}{\sqrt{2}}&-\frac{i}{\sqrt{2}}\\\frac{i}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{bmatrix}_{2\times2}$$

$$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(\Sigma_2+\Sigma_3))=\begin{bmatrix} 1 & -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 1 & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 & -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 1 & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 1 & -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \end{bmatrix}_{6\times 6}$$

rank [$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(\Sigma_2+\Sigma_3))$] = 6 , Inertia[$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(\Sigma_2+\Sigma_3))$] = {5,0,1}

eigenspectrum [$R(N=2\,|\,A=\frac{1}{\sqrt{2}}(\Sigma_2+\Sigma_3))$] = { 3.12132, 2.151801, 2, 0.292893, 0.292893, -1.858908 } .... (upto 6 decimal places)

IV. DISCUSSION AND CONCLUSION

The present article formulates the Reduced Description framework associated with the Matrix Shell Model Formalism. It illustrates concept through the numerical results associated with selected U(2) matrices. It can be observed that the $I_{2\times 2}$ and Σ_1 which differ from each other w.r.t repositioning of the same set of four numerical Matrix elements result in different associated R(N) matrices which share the same rank, Inertia and ordered eigenspectrum(ordered from largest to smallest along left to right). Similar thing is observed with Σ_2 and Σ_3 which possess the same value for rank, Inertia and ordered eigenspectrum. However, the $(I_{2\times 2},\Sigma_1)$ pair differs from the (Σ_2,Σ_3) in the value of Inertia and also the ordered eigenspectrum.

The Linear Combination $\frac{1}{\sqrt{2}}(I_{2\times 2}+i\Sigma_1)$ has the same rank, Inertia and ordered eigenspectrum as the $(I_{2\times 2},\Sigma_1)$, although the associated R(N) matrix is different, ($(I_{2\times 2},\Sigma_1,\frac{1}{\sqrt{2}}(I_{2\times 2}+i\Sigma_1))$ are associated with different, but Hermitian matrices of Positive semi definite type). However, this is not the case with $(\Sigma_2,\Sigma_3,\frac{1}{\sqrt{2}}(\Sigma_2+\Sigma_3))$, it can be observed that $\frac{1}{\sqrt{2}}(\Sigma_2+\Sigma_3)$ is associated with an R(N) matrix that is Hermitian, Indefinite and Invertible. Σ_2 and Σ_3 are associated with R(N) matrices that the Hermitian, Indefinite and Singular. The Hadamard Matrix of order 2, a linear combination of Σ_1 and Σ_3, is associated with rank, Inertia and ordered eigenspectrum completely different from that of Σ_1 and Σ_3. Further studies on other possible such Matrix subsets belonging to $M_{2\times 2}(C)$ and $M_{s\times s}(C)$, where $s>2$, will provide more depth of understanding on operation characteristics of the Reduced Matrix Shell Model framework and its applicability in solving real world problems.

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