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An Iterative Evolution Scheme on M (r, c) Subsets of Complex Matrix Spaces of Order m by n, where m≠n

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Abstract:- The article presents an iterative scheme for evolution on M(r,c) subsets of the complex matrix spaces $M_{m\times n}(C)$, where m≠n. The presented scheme involves separate conservation of the 'r' parameter and the 'c' parameter at each iteration step. The mathematical formalism is presented and illustrated with appropriate numerical examples.

Keywords:- *Global Mass Factor of a Matrix, Global Alignment Factor of a Matrix, M(r,c) Subsets of the Complex Matrix Spaces, Discrete Dynamical Systems, Markov Matrix, Hadamard Product of Matrices*

Notations

- $M_{m\times n}(C)$ denotes the Complex Matrix space of Matrices of order m by n
- $R(A)$ denotes the Global Mass Factor associated with the matrix $A_{m\times n}$
- $\hat{r}$ is the numerical realization of the $R(A)$ Factor
- $C(A)$ denotes the Global Alignment Factor associated with the matrix $A_{m\times n}$
- $\hat{c}$ is the numerical realization of the $C(A)$ Factor
- $|c|$ denotes the modulus of the complex number c
- $c^{\bullet}$ denotes the complex conjugate of the complex number c
- $\{|e_1\rangle, |e_2\rangle, ..., |e_m\rangle\}$ denotes the standard Orthonormal basis in C^m and $\{|f_1\rangle, |f_2\rangle, ..., |f_n\rangle\}$ denotes the standard Orthonormal basis in C^n
- $|m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{m\times 1}, |n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}, |V\rangle = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_s \end{bmatrix}_{s\times 1}, \langle V| = [v_1^{\bullet} \quad v_2^{\bullet} \quad . \quad . \quad v_s^{\bullet}]_{1\times s}$
- $B = [b_{ij}]_{m\times n}, \langle V|B|W\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij} v_i^{\bullet} w_j$
- X^H denotes the Hermitian conjugate of the matrix X
- X^T denotes the Transpose of the matrix X
- $M(\hat{r}, \hat{c})$ Is a subset of the Complex Matrix space $M_{m\times n}(C)$, characterized by the numerical values of the Global Mass factor and Global alignment factor, $\hat{r}$ and $\hat{c}$, respectively
- N denotes the set of all Natural numbers
- C denotes the set of all Complex numbers
- $U_{m\times n} \circ V_{m\times n}$ denotes the Hadamard Product[10,19,26] of the matrices U and V of order m by n
- $B_{m\times n} > 0$ implies $B = \sum_{x=1}^{m}\sum_{y=1}^{n} b_{xy} |e_x\rangle\langle f_y|$, B is real valued and $b_{xy} > 0, \ \forall \ x = 1,2,....,m$ and $y = 1,2,....,n$
- $B_{m\times n} \geq 0$ implies $B = \sum_{x=1}^{m}\sum_{y=1}^{n} b_{xy} |e_x\rangle\langle f_y|$, B is real valued and $b_{xy} \geq 0$, $\forall \ x = 1,2,....,m$ and $y = 1,2,....,n$

I. INTRODUCTION

The research article proposes a mathematical scheme for iterative evolution on M(r,c) subsets of strictly rectangular complex matrix spaces of order 'm' by 'n'. The conservation of the subset parameter 'r' is achieved in an iterative framework involving Markov matrices [5, 8, 12, 25] whose structure is dictated by the singular values of the Modulus matrix $\Sigma_{m\times n}$ of the present iteration. The Iterative evolution of the Phase matrix $\Phi_{m\times n}$ involves re-adjustment of the individual phase terms (the matrix elements of the Phase matrix) preserving their modulus to unity, and ensuring the conformance to the convention requirement: $r_{ij} = 0 \rightarrow c_{ij} = 1$, such that the 'c' parameter is conserved at every iteration step.

The article presents the mathematical formalism and illustrates the same with suitable numerical examples. A section on a hypothetical case which presents the numerical illustration of phase re-adjustment in presence of non-negativity but lack of positivity in $\Sigma_{m\times n}$ follows the set of illustrative examples. The article concludes with a discussion on the case studies and on observations pertaining to the mathematical framework associated with the evolution scheme.

II. MATHEMATICAL FRAMEWORK AND ASSOCIATED ANALYSIS

The following results, stated in [20], [21], [22] and [23] provide the groundwork for the formalism described in this article:

- $A \in M_{m\times n}(C)$, $A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} |e_i\rangle\langle f_j|$, $a_{ij} = r_{ij}c_{ij}$, $a_{ij} \neq 0 \mapsto r_{ij} = |a_{ij}|, c_{ij} \in C, |c_{ij}| = 1$,we consider the following convention in the case of zero matrix elements of matrix $A:\ a_{ij} = 0 \mapsto r_{ij} = 0, c_{ij} = 1$
- $R(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}$, $C(A) = c_{11}c_{12}....c_{1n}c_{21}c_{22}....c_{2n}.........c_{m1}c_{m2}....c_{mn} = \prod_{i=1}^{m}\prod_{j=1}^{n} c_{ij}$, we have therefore the following: $R(A) \geq 0,\ C(A) \in C, |C(A)| = 1, \forall A \in M_{m\times n}(C)$
- $M(\hat{r},\hat{c}) \subset M_{m\times n}(C), M(\hat{r},\hat{c}) = \{A \in M_{m\times n}(C) \mid A \neq 0_{m\times n}, R(A) = \hat{r}, C(A) = \hat{c}\}$, where we have the condition: $\hat{r} > 0, \hat{c} \in C, |\hat{c}| = 1$
- $\Sigma_{m\times n} = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij} |e_i\rangle\langle f_j|$, $\Phi_{m\times n} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} |e_i\rangle\langle f_j|$, we have: $A_{m\times n} = \Sigma_{m\times n} \circ \Phi_{m\times n}$

We define the following: $\Omega(t): M(\hat{r},\hat{c}) \mapsto M(\hat{r},\hat{c})$, $t = 0,1,2,....$, $t \in \{0\} \cup N$, we have the following recursion model:

$A_{m\times n}(t+1) = \Sigma_{m\times n}(t+1) \circ \Phi_{m\times n}(t+1) = \Omega(t)[A_{m\times n}(t)] = \Omega(t)[\Sigma_{m\times n}(t) \circ \Phi_{m\times n}(t)]$, with the Imposed initial condition: $A_{m\times n}(0) = \Sigma_{m\times n}(0) \circ \Phi_{m\times n}(0)$, $A_{m\times n}(0) \in M(\hat{r},\hat{c})$

Iterative scheme for $\Sigma_{m\times n}$:

- $\omega(t)$ denotes the rank of the matrix $\Sigma_{m\times n}(t)$
- singular values of $\Sigma_{m\times n}(t)$: $\sigma_1(t) \geq \sigma_2(t) \geq ....\sigma_{\omega(t)}(t) > 0$, we define: $\sigma_T(t) = \sum_{j=1}^{\omega(t)} \sigma_j(t)$
- We define the vectors $|p(t)\rangle$ and $|q(t)\rangle$ as follows:

$$|p(t)\rangle_{m\times 1} = \left[\frac{m\sigma_1(t)}{\sigma_T(t)[2m-\omega(t)]} \quad \frac{m\sigma_2(t)}{\sigma_T(t)[2m-\omega(t)]} \quad . \quad . \quad \frac{m\sigma_{\omega(t)}(t)}{\sigma_T(t)[2m-\omega(t)]} \quad \frac{1}{[2m-\omega(t)]} \quad \frac{1}{[2m-\omega(t)]} \quad . \quad . \quad \frac{1}{[2m-\omega(t)]}\right]^T$$

$$|q(t)\rangle_{n\times 1} = \left[\frac{n\sigma_1(t)}{\sigma_T(t)[2n-\omega(t)]} \quad \frac{n\sigma_2(t)}{\sigma_T(t)[2n-\omega(t)]} \quad . \quad . \quad \frac{n\sigma_{\omega(t)}(t)}{\sigma_T(t)[2n-\omega(t)]} \quad \frac{1}{[2n-\omega(t)]} \quad \frac{1}{[2n-\omega(t)]} \quad . \quad . \quad \frac{1}{[2n-\omega(t)]}\right]^T$$

- We define the following Markov type Matrices[5, 8, 12, 25] :

$$P_{m\times m}(t) = |p(t)\rangle\langle m| + \left(\frac{1}{m.n}\right)[I_{m\times m} - |p(t)\rangle\langle m|]$$

$$Q_{n\times n}(t) = |q(t)\rangle\langle n| + \left(\frac{1}{m.n}\right)[I_{n\times n} - |q(t)\rangle\langle n|]$$

- The Recursion model for $\Sigma_{m\times n}$ is given as follows:

$$\Sigma_{m\times n}(t+1) = P_{m\times m}(t)\Sigma_{m\times n}(t)Q_{n\times n}{}^T(t) \quad , \quad t = 0,1,2,..., \quad t \in \{0\} \cup N$$

Iterative scheme for $\Phi_{m\times n}$:

- We define the following matrices:

$S_{m\times m} = diag[1, \exp(+i(\frac{2\pi}{n})), ..., \exp(+i[m-1](\frac{2\pi}{n}))]$,

$T_{n\times n} = diag[1, \exp(+i(\frac{2\pi}{m})), ..., \exp(+i[n-1](\frac{2\pi}{m}))]$, in these two matrix descriptions, 'i' denotes the imaginary unit, i.e. $i^2 = -1$

- We define: $\bar{\Phi}_{m\times n}(t+1) = S_{m\times m}\Phi_{m\times n}(t)T_{n\times n}$, $t = 0,1,2,...,$ $t \in \{0\} \cup N$

Rule:

a. If $\Sigma_{m\times n}(t+1) > 0$, Then : $\Phi_{m\times n}(t+1) = \bar{\Phi}_{m\times n}(t+1)$

b. If $\Sigma_{m\times n}(t+1) \geq 0$, Then $\Phi_{m\times n}(t+1)$ is computed from $\bar{\Phi}_{m\times n}(t+1)$ through the following sequence of steps:

1. Compute the Matrices $X_{m\times n}(t+1)$ and $\bar{X}_{m\times n}(t+1)$ as follows:

$X_{m\times n}(t+1) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}(t+1)\left|e_i\right\rangle\left\langle f_j\right|$, where $x_{ij}(t+1) = 1$ when $r_{ij}(t+1) \neq 0$ and $x_{ij}(t+1) = 0$ when $r_{ij}(t+1) = 0$, $i = 1,2,...,m$, $j = 1,2,....,n$

$\bar{X}_{m\times n}(t+1) = (\left|m\right\rangle\left\langle n\right|) - X_{m\times n}(t+1)$

2. $\ddot{\Phi}_{m\times n}(t+1) = X_{m\times n}(t+1) \circ \bar{\Phi}_{m\times n}(t+1)$, $\hat{\Phi}_{m\times n}(t+1) = \bar{X}_{m\times n}(t+1) + \ddot{\Phi}_{m\times n}(t+1)$, $\hat{\Phi}_{m\times n}(t+1) = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{c}_{ij}(t+1)\left|e_i\right\rangle\left\langle f_j\right|$

3. $\bar{c}(t+1) = \prod_{x=1}^{m}\prod_{y=1}^{n} \bar{c}_{xy}(t+1)$, $\rho(t+1) = \bar{c}(t+1)^{\bullet}\hat{c}$, 'i' denotes the imaginary unit, i.e. $i^2 = -1$ $\rho(t+1) = \exp[+i\eta(t+1)]$ where $\eta(t+1) \in [0, 2\pi)$ $\forall$ $t = 0,1,2,...,$ $t \in \{0\} \cup N$

4. $\delta(t+1)$ denotes the number of zero elements in $\Sigma_{m\times n}(t+1)$. 'i' denotes the imaginary unit, i.e. $i^2 = -1$, We define the following: $\varepsilon(t+1) = \exp[+i(\frac{\eta(t+1)}{m.n - \delta(t+1)})]$

5. $\hat{\Delta}_{m\times n}(t+1) = [\varepsilon(t+1)\left|m\right\rangle\left\langle n\right|]_{m\times n} \circ X_{m\times n}(t+1)$, $\Delta_{m\times n}(t+1) = \bar{X}_{m\times n}(t+1) + \hat{\Delta}_{m\times n}(t+1)$

6. $\Phi_{m\times n}(t+1) = \Delta_{m\times n}(t+1) \circ \hat{\Phi}_{m\times n}(t+1) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}(t+1)\left|e_i\right\rangle\left\langle f_j\right|$

We have the following conservation equations associated with the Iterative evolution scheme:

- $\sum_{x=1}^{m}\sum_{y=1}^{n} r_{xy}(t) = \hat{r}$, $t = 0,1,2,....$, $t \in \{0\} \cup N$
- $\prod_{x=1}^{m}\prod_{y=1}^{n} c_{xy}(t) = \hat{c}$, $t = 0,1,2,....$, $t \in \{0\} \cup N$

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Numerical Examples

1) $A_{2\times3}(0)=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}_{2\times3}$, We have : $A\in M(\hat{r}=2,\hat{c}=1)$

We have the following set of Results:

- $\Sigma_{2\times3}(0)=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}_{2\times3}$, $\Phi_{2\times3}(0)=\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}_{2\times3}$

- $P_{2\times2}(0)=\left(\frac{1}{12}\right)\begin{bmatrix}7&5\\5&7\end{bmatrix}$, $Q_{3\times3}(0)=\left(\frac{1}{48}\right)\begin{bmatrix}23&15&15\\15&23&15\\10&10&18\end{bmatrix}$

- $\Sigma_{2\times3}(1)=\left(\frac{1}{144}\right)\begin{bmatrix}59&55&30\\55&59&30\end{bmatrix}_{2\times3}$, $\Phi_{2\times3}(1)=\begin{bmatrix}1&-1&1\\\phi&-\phi&\phi\end{bmatrix}_{2\times3}$, where $\phi=-\left(\frac{1}{2}\right)+i\left(\frac{\sqrt{3}}{2}\right)$, $i^2=-1$

- $A_{2\times3}(1)=\begin{bmatrix}0.409722&-0.381944&0.208333\\-0.190972+0.330774i&0.204861-0.354830i&-0.104167+0.180422i\end{bmatrix}_{2\times3}$ .... (up to 6 decimal places)

- $P_{2\times2}(1)=\begin{bmatrix}0.973469&0.806802\\0.026531&0.193198\end{bmatrix}$, $Q_{3\times3}(1)=\begin{bmatrix}0.771768&0.605102&0.605102\\0.019898&0.186565&0.019898\\0.208333&0.208333&0.375\end{bmatrix}$

.... (up to 6 decimal places)

- $\Sigma_{2\times3}(2)=\begin{bmatrix}1.195079&0.152487&0.432705\\0.147068&0.019254&0.053406\end{bmatrix}_{2\times3}$ ....(up to 6 decimal places)

- $\Phi_{2\times3}(2)=\begin{bmatrix}1&1&1\\\phi^{\bullet}&\phi^{\bullet}&\phi^{\bullet}\end{bmatrix}_{2\times3}$

- $A_{2\times3}(2)=\begin{bmatrix}1.195079&0.152487&0.432705\\-0.073534-0.127365i&-0.009627-0.016674i&-0.026703-0.046251i\end{bmatrix}_{2\times3}$ .... (up to 6 decimal places)

2) $B_{2\times3}(0)=\begin{bmatrix}2&0&0\\0&0&0\end{bmatrix}_{2\times3}$, We have : $B\in M(\hat{r}=2,\hat{c}=1)$

We have the following set of Results:

- $\Sigma_{2\times3}(0)=\begin{bmatrix}2&0&0\\0&0&0\end{bmatrix}_{2\times3}$, $\Phi_{2\times3}(0)=\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}_{2\times3}$

- $P_{2\times2}(0)=\left(\frac{1}{18}\right)\begin{bmatrix}13&10\\5&8\end{bmatrix}$, $Q_{3\times3}(0)=\left(\frac{1}{6}\right)\begin{bmatrix}4&3&3\\1&2&1\\1&1&2\end{bmatrix}$

- $\Sigma_{2\times3}(1)=\left(\frac{1}{108}\right)\begin{bmatrix}104 & 26 & 26\\ 40 & 10 & 10\end{bmatrix}_{2\times3}$, $\Phi_{2\times3}(1)=\begin{bmatrix}1 & -1 & 1\\ \phi & -\phi & \phi\end{bmatrix}_{2\times3}$

- $B_{2\times3}(1)=\begin{bmatrix}0.962963 & -0.240741 & 0.240741\\ -0.185185+0.320750i & 0.046296-0.080188i & -0.046296+0.080188i\end{bmatrix}_{2\times3}$

.... (up to 6 decimal places)

- $P_{2\times2}(1)=\left(\frac{1}{18}\right)\begin{bmatrix}13 & 10\\ 5 & 8\end{bmatrix}$, $Q_{3\times3}(1)=\left(\frac{1}{6}\right)\begin{bmatrix}4 & 3 & 3\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$

- $\Sigma_{2\times3}(2)=\begin{bmatrix}0.826132 & 0.262860 & 0.262860\\ 0.396091 & 0.126029 & 0.126029\end{bmatrix}_{2\times3}$ .... (up to 6 decimal places)

- $\Phi_{2\times3}(2)=\begin{bmatrix}1 & 1 & 1\\ \phi^{\bullet} & \phi^{\bullet} & \phi^{\bullet}\end{bmatrix}_{2\times3}$

- $B_{2\times3}(2)=\begin{bmatrix}0.826132 & 0.262860 & 0.262860\\ -0.198045-0.343024i & -0.063014-0.109144i & -0.063014-0.109144i\end{bmatrix}_{2\times3}$ .... (up to 6 decimal places)

3) $D_{2\times3}(0)=\begin{bmatrix}-i & 0 & 0\\ 0 & +i & 0\end{bmatrix}_{2\times3}$, We have: $D\in M(\hat{r}=2,\hat{c}=1)$

We have the following set of Results:

- $\Sigma_{2\times3}(0)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}_{2\times3}$, $\Phi_{2\times3}(0)=\begin{bmatrix}-i & 1 & 1\\ 1 & +i & 1\end{bmatrix}_{2\times3}$

- $P_{2\times2}(0)=\left(\frac{1}{12}\right)\begin{bmatrix}7 & 5\\ 5 & 7\end{bmatrix}$, $Q_{3\times3}(0)=\left(\frac{1}{48}\right)\begin{bmatrix}23 & 15 & 15\\ 15 & 23 & 15\\ 10 & 10 & 18\end{bmatrix}$

- $\Sigma_{2\times3}(1)=\left(\frac{1}{144}\right)\begin{bmatrix}59 & 55 & 30\\ 55 & 59 & 30\end{bmatrix}_{2\times3}$, $\Phi_{2\times3}(1)=\begin{bmatrix}-i & -1 & 1\\ \phi & \chi & \phi\end{bmatrix}_{2\times3}$, where $\chi=\left(\sqrt{3}/2\right)+i\left(1/2\right)$, $i^2=-1$

- $D_{2\times3}(1)=\begin{bmatrix}-0.409722i & -0.381944 & 0.208333\\ -0.190972+0.330774i & 0.354830+0.204861i & -0.104167+0.180422i\end{bmatrix}_{2\times3}$

.... (up to 6 decimal places)

- $P_{2\times2}(1)=\begin{bmatrix}0.973469 & 0.806802\\ 0.026531 & 0.193198\end{bmatrix}$, $Q_{3\times3}(1)=\begin{bmatrix}0.771768 & 0.605102 & 0.605102\\ 0.019898 & 0.186565 & 0.019898\\ 0.208333 & 0.208333 & 0.375\end{bmatrix}$

- $\Sigma_{2\times3}(2) = \begin{bmatrix} 1.195079 & 0.152487 & 0.432705 \\ 0.147068 & 0.019254 & 0.053406 \end{bmatrix}_{2\times3}$ .... (up to 6 decimal places)

- $\Phi_{2\times3}(2) = \begin{bmatrix} -i & 1 & 1 \\ \phi^{\bullet} & \chi\bullet & \phi^{\bullet} \end{bmatrix}_{2\times3}$

- $D_{2\times3}(2) = \begin{bmatrix} -1.195079i & 0.152487 & 0.432705 \\ -0.073534-0.127365i & 0.016674-0.009627i & -0.026703-0.046251i \end{bmatrix}_{2\times3}$ .... (up to 6 decimal places)

<u>Numerical Example of Phase redistribution scheme when associated R matrix is non-negative and not strictly positive</u>

- $\Sigma_{2\times3}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{2\times3}$, $\bar{\Phi}_{2\times3}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & +i \end{bmatrix}_{2\times3}$, $A_{2\times3}(t) \in M(\hat{r}=5, \hat{c}=+i)$, $t \in \{0\} \cup N$

- $X(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{2\times3}$, $\bar{X}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{2\times3}$

- $\hat{\Phi}_{2\times3}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times3}$, $\bar{c}(t) = 1$, $\rho(t) = +i$, $\eta(t) = \pi/2$, $(m.n - \delta(t)) = 5$, $\varepsilon(t) = \exp[+i(\pi/10)]$

- $\Delta_{2\times3}(t) = \begin{bmatrix} \psi & \psi & \psi \\ \psi & \psi & 1 \end{bmatrix}_{2\times3}$, where $\psi = \exp[+i(\pi/10)]$

- $\Phi_{2\times3}(t) = \begin{bmatrix} \psi & \psi & \psi \\ \psi & \psi & 1 \end{bmatrix}_{2\times3}$, $A_{2\times3}(t) = \begin{bmatrix} \psi & \psi & \psi \\ \psi & \psi & 0 \end{bmatrix}_{2\times3}$

III. DISCUSSION AND CONCLUSION

The Iterative evolution scheme presented in the article describes a discrete evolution scheme [4,6] on M(r,c) subsets of strictly rectangular (i.e. m≠n) complex matrix spaces of order m by n. The real life problems in science and engineering, which can be mapped onto M(r,c) subsets of complex matrix spaces, the presented scheme provides for those problems a possible framework to describe a discrete dynamical/discrete evolutionary system on such subsets.

In the Illustrative examples presented in the article, the matrices $A_{2\times3}(0)$, $B_{2\times3}(0)$ and $D_{2\times3}(0)$ belong to $M(\hat{r}=2, \hat{c}=1)$ subset of $M_{2\times3}(C)$, they represent different Initial conditions for the Iteration framework and hence, different evolutionary routes in the Matrix space confined on the particular M(r,c) subset. $A_{2\times3}(0)$ and $D_{2\times3}(0)$ have the same Modulus matrix ($\Sigma_{m\times n}$) component but different phase matrix component ($\Phi_{m\times n}$), hence, conservation of the 'r' parameter occurs along the same iterative route for these initial conditions, but conservation of the 'c' parameter follows along different routes, this ultimately results in a different overall iterative evolution being associated with these two initial conditions. Similarly, $A_{2\times3}(0)$ and $B_{2\times3}(0)$ have the same phase matrix component but different Modulus matrix component, therefore it results in different iterative evolution routes for conservation of the 'r' parameter and different overall iterative evolution.

In the Hypothetical example, the modulus matrix associated with the particular iteration step is non -negative but not strictly positive, hence, the conformance to the convention $r_{ij} = 0 \to c_{ij} = 1$ require readjustment and redistribution of the phase terms. The matrices $\Sigma_{2\times3}(t)$ and $\Phi_{2\times3}(t)$ satisfy the imposed convention and generate the $A_{2\times3}(t)$ matrix.

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Deeper understandings of the iterative evolution scheme require analysis of the limiting behavior of the scheme and its dependence, if any, on the initial conditions. These issues will be addressed in subsequent follow up studies.

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