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On Bayesian Estimation of Fuction of Unknown Parameter of Modified Power Series Distribution

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Abstract:- This paper deals with the Bayesian estimation of a function of the unknown parameter θ of Modified Power Series distribution. These estimates have similar forms as the classical MVUE given by Gupta (1977), but are better than MVUV in the sense ,that they increase the range of estimation.The prior distribution for the unknown parameter θ varies from distribution to distribution, depending upon the range of θ.On the part of loss functions, the Squared Error Loss Function (SELF) and two different forms of Weighted Squared Error Loss Function (WSELF) has been considered.

***Keywords**:- Modified Power Series Distribution, Bayes Estimator, Squared Error Loss Function, Weighted Squared Error Loss Function.*

I. INTRODUCTION

A discrete random variable X is said to have Modified Power Series distribution, if its probability mass function (p. m .f.) $p_\theta(x) = P(X = x)$ is given by,

$$p_\theta(x) = \begin{cases} \frac{a(x)\{g(\theta)\}^x}{f(\theta)}, \text{if } x \in S, \theta \in A \\ 0, \text{Otherwise.} \end{cases} \quad (1)$$

Where, θ is unknown parameter of the distribution,$A \subseteq \mathcal{R}$ (the set of real numbers), $a(x) > 0, S$ is a subset of the set of non-negative integers, $g(\theta) > 0$ and $f(\theta)$ is a function of θ such that $\sum_{x\in S} p_\theta(x) = f(\theta)$

As mentioned by Gupta (1974) the p. m .f. given by (1) covers a wide range of discrete distributions. When $g(\theta) = \theta$, (1) coincides with the class of discrete distributions as given by Roy and Mitra (1957).

Gupta (1977), has obtained MVUE of $\phi(\theta) = \theta^r, r \geq 1$.For values of $r < 1$,no unbiased estimator of $\phi(\theta)$ exists and hence no MVUE of $\phi(\theta)$ exists. This is a serious limitation of this Classical estimator. In this paper, Bayes Estimator of $\phi(\theta) = \theta^r, r \in (-\infty, \infty)$.Here the range of estimation is increased as we have taken $r \in (-\infty, \infty)$.

On the part of loss functions, the usual Squared Error Loss Function (SELF)and two different forms of the Weighted Squared Error Loss Function (WSELF) have been taken.

II. NOTATIONS AND RESULTS USED:

Let $X_1, X_2, X_3, \ldots X_N$ be a random sample of size N from the p .m. f given by (1).
Then,
$$T_N = \sum_{i=1}^N X_i \quad (2)$$

We shall use the following result as given by Abramowitz and Stegun (1964):
$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du \quad (3)$$
$$\Gamma(x) b^{-x} = \int_0^\infty u^{x-1} e^{-bu} du \quad (4)$$
$$\frac{\Gamma(b-a)\Gamma(a)M(a,b,z)}{\Gamma(b)} = \int_0^1 u^{a-1}(1-u)^{b-a-1} e^{-zu} du \quad (5)$$

Where, $M(a, b, z)$ is the Confluent Hypergeometric Function and has a series representation given by,
$$M(a, b, z) = \sum_{n=0}^\infty \frac{(a)_n z^n}{(b)_n n!} \quad (6)$$
Where, $(a)_0 = 1$ amd
$$(a)_n = \prod_{i=1}^n (a + i - 1) \quad (7)$$

For observed value $t_N = \sum_{i=1}^N x_i$ of the statistic $T_N = \sum_{i=1}^N X_i$, the likelihood function, denoted by $L(\theta)$, is given by,
$$L(\theta) = k\{g(\theta)\}^{t_N}\{f(\theta)\}^{-N} \quad (8)$$
Where, k is function of $x_1, x_2, x_3, \ldots x_N$ and does not contain θ.
Let $\pi(\theta)$ be the prior probability density function of θ,then the posterior posterior probability density function of θ ,denoted by $\pi(\theta / t_N)$,is given by,
$$\pi(\theta / t_N) = \frac{L(\theta)\pi(\theta)}{\int_A L(\theta)\pi(\theta) d\theta} \quad (9)$$

Under the Squared Error Loss Function (SELF), $L(\phi(\theta), d) = (\phi(\theta) - d)^2$,the Bayes Estimate of $\phi(\theta)$,denoted by $\hat{\phi}_B$ is given by,
$$\hat{\phi}_B = \int_A \phi(\theta)\pi(\theta / t_N) d\theta \quad (10)$$

Similarly, under the Weighted Squared Error Loss Function (WSELF), $L(\phi(\theta), d) = W(\theta)(\phi(\theta) - d)^2$, where, $W(\theta)$ is a function of θ, the Bayes Estimate of $\phi(\theta)$,denoted by $\hat{\phi}_W$ is given by,
$$\hat{\phi}_W = \frac{\int_A W(\theta)\phi(\theta)\pi(\theta / t_N) d\theta}{\int_A W(\theta)\pi(\theta / t_N) d\theta} \quad (11)$$

We have taken two different forms of $W(\theta)$,as given below:
(i). $W(\theta) = \theta^{-2}$.The Bayes Estimate of $\phi(\theta)$,denoted by $\hat{\phi}_M$, is known as the Minimum Expected Loss (MELO) Estimate and is given by,

$$\hat{\phi}_M = \frac{\int_A \theta^{-2}\phi(\theta)\pi(\theta/t_N)d\theta}{\int_A \theta^{-2}\pi(\theta/t_N)d\theta} \qquad (12)$$

This loss function was used by Tummala and Sathe (1978) for estimating reliability of certain life time distributions and by Zellner (1979) for estimating functions of parameters in econometric models.
(ii). $W(\theta) = \theta^{-2}e^{-a\theta^{-1}}$.The Bayes Estimate of $\phi(\theta)$,denoted by $\hat{\phi}_E$, is known as the Exponentially Minimum Expected Loss EW(MELO) Estimate and is given by,

$$\hat{\phi}_E = \frac{\int_A \theta^{-2}e^{-a\theta^{-1}}\phi(\theta)\pi(\theta/t_N)d\theta}{\int_A \theta^{-2}e^{-a\theta^{-1}}\pi(\theta/t_N)d\theta} \qquad (13)$$

This type of loss function was used by the author (1977) for the first time in his work for D.Phil.
Now, we shall some special cases of the p. m. f. given by (1) and obtain corresponding Bayes Estimate of $\phi(\theta) = \theta^r, r \in (-\infty,\infty)$, in each case.

III. GENERALIZED NEGATIVE BINOMIAL DISTRIBUTION (GNBD)

If we take $a(x) = \frac{n\Gamma(n+\beta x)}{\Gamma(x+1)\Gamma(n+\beta x-x+1)}$, $g(\theta) = \theta(1-\theta)^{\beta-1}$, $f(\theta) = (1-\theta)^{-n}$, $S = \{0,1,2\ldots\infty\}$, $A = (0,1)$, $\beta \geq 0$, $\theta\beta \in (-1,1)$, n being a positive integer, the corresponding discrete random variable X is said to have Generalized Negative Binomial distribution.

In this case,

$$L(\theta) = k\theta^{t_N}(1-\theta)^{t_N(\beta-1)+nN} \qquad (14)$$

Since, in this case, $0 < \theta < 1$, we have taken two different prior distributions, namely, $\pi_1(\theta)$ and $\pi_2(\theta)$ as given below:

$$\pi_1(\theta) = \begin{cases} \frac{\theta^{p-1}(1-\theta)^{q-1}}{B(p,q)}, \text{if } p > 0, q > 0, 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \qquad (15)$$

And,

$$\pi_2(\theta) = \begin{cases} \frac{e^{-b\theta}\theta^{p-1}(1-\theta)^{q-1}}{B(p,q)M(p,p+q,-b)}, \text{if } p > 0, q > 0, 0 < \theta < 1, b \geq 0 \\ 0, \text{Otherwise.} \end{cases}$$

(16)

Where,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (17)$$

The posterior p. d .f. of θ , corresponding to the prior $\pi_1(\theta)$, denoted by $\pi_1(\theta/t_N)$,is given by,

$$\pi_1(\theta/t_N) = \begin{cases} \frac{\theta^{t_N+p-1}(1-\theta)^{t_N(\beta-1)+nN+q-1}}{B(t_N+p,(\beta-1)t_N+nN+q)}, \text{if } p > 0, q > 0, 0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases}$$

(18)

Similarly, posterior p. d .f. of θ , corresponding to the prior $\pi_2(\theta)$, denoted by $\pi_2(\theta/t_N)$,is given by

$$\pi_2(\theta/t_N) = \begin{cases} \frac{e^{-b\theta}\theta^{t_N+p-1}(1-\theta)^{t_N(\beta-1)+nN+q-1}}{K}, \text{if } p > 0, q > 0, 0 < \theta < 1, b \\ 0, \text{Otherwise.} \end{cases}$$

Where,

$$K = B(t_N+p,(\beta-1)t_N+nN+q)M(t_N+p,p+q+\beta t_N+nN,-b) \qquad (20)$$

Under the SELF and corresponding to the posterior distribution given by (18), Bayes Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_{1B}^t$ is given by,

$$\hat{\theta}_{1B}^t = \frac{B(t_N+p+r,(\beta-1)t_N+nN+q)}{B(t_N+p,(\beta-1)t_N+nN+q)} \qquad (21)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (18), the MELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_{1M}^t$ is given by,

$$\hat{\theta}_{1M}^t = \frac{B(t_N+p+r-2,(\beta-1)t_N+nN+q)}{B(t_N+p-2,(\beta-1)t_N+nN+q)} \qquad (22)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta^{-1}}$ and corresponding to the posterior distribution given by (18), the EWMELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_{1E}^t$ is given by,

$$\hat{\theta}_{1E}^t = \frac{B(t_N+p+r-2,(\beta-1)t_N+nN+q)M_2}{B(t_N+p-2,(\beta-1)t_N+nN+q)M_1} \qquad (23)$$

Where,

$$M_1 = M(t_N+p-2,p+q+\beta t_N+nN-2,-a)$$

(24)

$$M_2 = M(t_N+p+r-2,p+q+\beta t_N+nN+r-2,-a)$$

(25)

On the other hand, under the SELF and corresponding to the posterior distribution given by (19), Bayes Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_{2B}^t$ is given by,

$$\hat{\theta}_{2B}^t = \frac{B(t_N+p+r,(\beta-1)t_N+nN+q)M_4}{B(t_N+p,(\beta-1)t_N+nN+q)M_3} \qquad (26)$$

Where,

$$M_3 = M(t_N+p,p+q+\beta t_N+nN,-b)$$

(27)

$$M_4 = M(t_N+p+r,p+q+\beta t_N+nN+r,-b)$$

(28)

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (19), the MELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_{2M}^t$ is given by,

$$\hat{\theta}_{2M}^t = \frac{B(t_N+p+r-2,(\beta-1)t_N+nN+q)M_6}{B(t_N+p-2,(\beta-1)t_N+nN+q)M_5} \qquad (29)$$

Where,

$$M_5 = M(t_N+p-2,p+q+\beta t_N+nN-2,-b)$$

(30)

$$M_6 = M(t_N+p+r-2,p+q+\beta t_N+nN+r-2,-b)$$

(31)

Finally, under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta^{-1}}$ and corresponding to the posterior distribution given by (19),the EWMELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}^t_{2E}$ is given by,

$$\hat{\theta}^t_{2E} = \frac{B(t_N+p+r-2,(\beta-1)t_N+nN+q)M_8}{B(t_N+p-2,(\beta-1)t_N+nN+q)M_7} \quad (32)$$

Where,

$$M_7 = M(t_N + p - 2, p + q + \beta t_N + nN - 2, -(a + b)) \quad (33)$$

$$M_2 = M(t_N + p + r - 2, p + q + \beta t_N + nN + r - 2, -(a + b)) \quad (34)$$

Remark (1): The MVUE of θ^r is 0 if $z < r$ $(z = t_N)$ which is a serious limitation of the MVUE, in this case. The Bayes Estimates on the other hand, are 0 if $r < 0$ is such that $t_N + p < -r, p + q + \beta t_N + nN < -r, t_N + p - 2 < -r, p + q + \beta t_N + nN - 2 < -r$ depending on various loss functions and two posterior distributions.

SPECIAL CASE: Since, for $\beta = 1$,the GNBD coincides with the Negative Binomial Distribution (NBD), all results as derived above give, Bayes Estimate of θ^r for the NBD when $\beta = 1$. Additionally ,when $\beta = 1$ and n=1,we get Bayes Estimate of θ^r for the Geometric distribution. When $\beta = 0$, we get Bayes Estimate of θ^r for the Binomial distribution.

IV. GENERALIZED LOGARITHMIC SERIES DISTRIBUTION (GLSD)

If we take $a(x) = \frac{\Gamma(\beta x)}{\Gamma(x+1)\Gamma(\beta x-x+1)}, g(\theta) = \theta(1-\theta)^{\beta-1}, f(\theta) = -\ln(1-\theta)$, $S = \{1,2 \ldots \infty\}, A = (0,1), \beta \geq 0, \theta\beta \in (0,1)$,n being a positive integer, the corresponding discrete random variable X is said to have Generalized Logarithmic Series distribution.

Since in this case, $0 < \theta < 1$,we take $\pi_3(\theta)$ as the p. d. f. of Negative Log Gamma distribution given by

$$\pi_3(\theta) = \begin{cases} \frac{(k+1)^{N+1}(1-\theta)^k\{-\ln(1-\theta)\}^N}{\Gamma(N+1)}, \text{if } k > 0,,0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (35)$$

Where, N, a positive integer is same as the size of the random sample.

The posterior p. d .f. of θ ,denoted by $\pi_3(\theta / t_N)$, is given by

$$\pi_3(\theta / t_N) = \begin{cases} \frac{\theta^{t_N}(1-\theta)^{t_N(\beta-1)+k}}{B(t_N+1,(\beta-1)t_N+k+1)}, \text{if } k > 0,,0 < \theta < 1 \\ 0, \text{Otherwise.} \end{cases} \quad (36)$$

Under the SELF and corresponding to the posterior distribution given by (36), Bayes Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}^t_B$ is given by,

$$\hat{\theta}^t_B = \frac{B(t_N+r+1,(\beta-1)t_N+k+1)}{B(t_N+1,(\beta-1)t_N+k+1)} \quad (37)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (36), the MELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}^t_M$ is given by,

$$\hat{\theta}^t_M = \frac{B(t_N+r-1,(\beta-1)t_N+k+1)}{B(t_N-1,(\beta-1)t_N+k+1)} \quad (38)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta^{-1}}$ and corresponding to the posterior distribution given by (36), the EWMELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}^t_E$ is given by,

$$\hat{\theta}^t_E = \frac{B(t_N+r-1,(\beta-1)t_N+k+1)M_{10}}{B(t_N-1,(\beta-1)t_N+k+1)M_9} \quad (39)$$

Where,

$$M_9 = M(t_N - 1, \beta t_N + k, -a) \quad (40)$$

$$M_{10} = M(t_N + r - 1, \beta t_N + r + k, -a) \quad (41)$$

Remark (2): The MVUE of θ^r is 0 if $z < r$, $(z = t_N)$ which is a serious limitation of the MVUE, in this case. The Bayes Estimates on the other hand, are 0 if $r < 0$ is such that $t_N + 1 < -r, t_N - 1 < -r$ $\beta t_N + k < -r$, depending on various loss functions.

SPECIAL CASE: Since, for $\beta = 1$,the GLSD coincides with the Logarithmic Series Distribution (LSD), all results as derived above give, Bayes Estimate of θ^r for the LSD when $\beta = 1$.

V. GENERALIZED POISSON DISTRIBUTION (GPD)

If we take $a(x) = \frac{\lambda_1(\lambda_1+\lambda_2 x)^{x-1}}{\Gamma(x+1)}, g(\theta) = \theta e^{-\theta\lambda_2}, f(\theta) = e^{-\theta\lambda_1}$, $S = \{0,1,2 \ldots \infty\}, A = (0, \infty),, \lambda_2 \in (-1,1)$, $\lambda_1 > 0$,the corresponding discrete random variable X is said to have Generalized Poisson distribution.

Since in this case, $\theta > 0$,we take $\pi_4(\theta)$ as the p. d. f. of Gamma distribution given by

$$\pi_4(\theta) = \begin{cases} \frac{c^\alpha \theta^{\alpha-1}e^{-c\theta}}{\Gamma(\alpha)}, \text{if } c > 0, \alpha < 0, \theta > 0 \\ 0, \text{Otherwise.} \end{cases} \quad (42)$$

The posterior p. d .f. of θ ,denoted by $\pi_4(\theta / t_N)$, is given by

$$\pi_4(\theta / t_N) = \begin{cases} \frac{(\lambda_2 t_N+N\lambda_1+c)^{t_N+\alpha}\theta^{t_N+\alpha-1}e^{-\theta(\lambda_2 t_N+N\lambda_1+c)}}{\Gamma(t_N+\alpha)}, \text{if } \theta > 0 \\ 0, \text{Otherwise.} \end{cases} \quad (43)$$

Under the SELF and corresponding to the posterior distribution given by (43), Bayes Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}^t_B$ is given by,

$$\hat{\theta}^t_B = \frac{\Gamma(t_N+\alpha+r)}{\Gamma(t_N+\alpha)(\lambda_2 t_N+N\lambda_1+c)^r} \quad (44)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (43), the MELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_M^t$ is given by,

$$\hat{\theta}_M^t = \frac{\Gamma(t_N+\alpha+r-2)}{\Gamma(t_N+\alpha-2)(\lambda_2 t_N+N\lambda_1+c)^r} \quad (45)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta^{-1}}$ and corresponding to the posterior distribution given by (43), the EWMELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_E^t$ is given by,

$$\hat{\theta}_E^t = \frac{\Gamma(t_N+\alpha+r-2)}{\Gamma(t_N+\alpha-2)(\lambda_2 t_N+N\lambda_1+c+a)^r} \quad (46)$$

Remark (3): The MVUE of θ^r exists as long $z \geq r$, $(z = t_N)$ and is 0 if $z < r$ which is a serious limitation of the MVUE, in this case. The Bayes Estimates on the other hand, are free from such restrictions between t_N and r as far as $r \geq 1$. This is another advantage of Bayesian Estimation over MVUE. However, if $r < 0$,Bayes Estimates are 0, if $t_N + \alpha < -r$ in (44) and $t_N + \alpha - 2 < -r$, in (45) and (46) respectively depending on various loss functions.

SPECIAL CASE: Since, for$\lambda_1 = 1$ and $\lambda_2 = 0$,the GPD coincides with the Poisson Distribution. So putting $\lambda_1 = 1$ and $\lambda_2 = 0$, equations (44),(45) and (46) respectively give, Bayes Estimate of θ^r for the Poisson Distribution as follows:.

$$\hat{\theta}_B^t = \frac{\Gamma(t_N+\alpha+r)}{\Gamma(t_N+\alpha)(N+c)^r} \quad (47)$$

Similarly, under the WSELF, when $W(\theta) = \theta^{-2}$ and corresponding to the posterior distribution given by (43), the MELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_M^t$ is given by,

$$\hat{\theta}_M^t = \frac{\Gamma(t_N+\alpha+r-2)}{\Gamma(t_N+\alpha-2)(N+c)^r} \quad (48)$$

Under the WSELF, when $W(\theta) = \theta^{-2}e^{-a\theta^{-1}}$ and corresponding to the posterior distribution given by (43), the EWMELO Estimate of $\phi(\theta) = \theta^r$,denoted by $\hat{\theta}_E^t$ is given by,

$$\hat{\theta}_E^t = \frac{\Gamma(t_N+\alpha+r-2)}{\Gamma(t_N+\alpha-2)(N+c+a)^r} \quad (49)$$

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