ISSN No:-2456-2165

Solving Linear and Nonlinear Singular Differential Equations Using Bessel Matrix Method

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Abstract:- We present a numerical method for obtaining an approximate polynomial solution for nonlinear singular differential equations using the Bessel matrix method based on collocation points. This method provides a system of linear equations with unknown Bessel coefficients. In addition, it presents high accuracy of solutions when exact solutions are polynomials. The reasoning and efficiency of the method are demonstrated via numerical examples. In addition, the method is compared with other techniques.

2020 Mathematics Subject Classification (MSC2020): 34B16; 34B05.

***Keywords**:- Bessel Matrix, Nonlinear, Singular, Numerical Solutions.*

I. INTRODUCTION

Nonlinear singular differential equations are used in several areas of mathematics and science (social and natural). Numerous mathematicians have studied differential equations, including Newton, Leibniz, the Bernoulli family, Riccati, Clairaut, d'Alembert, and Euler. Several methods have been developed to solve this class of differential equations and also higher order differential equations; for example, the sinc-Galerkin method [18], Chebyshev polynomials [6], Chebfun solution [6], cubic splines [5, 17], Adomian decomposition method and modified decomposition method [3], B-spline method [1], and differential transform method [2].

Based on collocation method and matrix relations between the Bessel polynomials and their derivatives, we use the Bessel matrix method for solving nonlinear singular differential equations with variable coefficients of the following form:

$$y^{(2r)}(x) + \frac{k_1}{x}y'(x) + \frac{k_2}{x^2}\,[y(x)]^{\nu} = g(x), \quad 0 \le x \le 1, \qquad (1.1)$$

subject to the following boundary conditions:

$$y^{(j)}(0) = \lambda_{0j}, \quad j = 2,3,\cdots,r-1,$$

$$y^{(j)}(1) = \lambda_{1j}, \quad j = 0,1,\cdots,r-1, \qquad (1.2)$$

where known functions $k_1(x)$, $k_2(x)$, and $g(x)$ are defined on $0 \le x \le 1$, ν is an integer constant, and λ_{0j} and λ_{1j} are real or complex constants.

The Bessel matrix method is an accurate and fast numerical solution method. The method is efficient and is proved by many researchers. It has been used to solve generalized pantograph equations [15], the systems of linear Fredholm integrodifferential equations [20], high-order linear Volterra integro-differential equations [16], a model of pollution for a system of lakes [21], the Bagley–Torvik equation, a class of Lane-Emden differential equations [13], a model for the HIV infection of CD4 $^{+}$Tcells [19], a fractional-order logistic population model, the first kind for singular perturbated differential equations, and residual correction.

The study is organized as follows: In Section 2, we discuss the Bessel function preliminaries and matrix relations used in our method. In Section 3, we show how the Bessel Matrix method used to have an equivalent explicit system of singular algebraic equations instead of equation (1.1). In addition, the boundary conditions for the equations are presented. In Section 4, we illustrate how the Bessel matrix method is used to solve a nonlinear higher-order boundary value problem (BVP). In Section 5, we present a few numerical examples and their exact solutions. Finally, we provide conclusion of the study in Section 6.

II. BESSEL FUNCTION PRELIMINARIES

Our aim is to find an approximate solution of equation (1.1) expressed in the truncated Bessel series form given by

$$y(x) = \sum_{n=0}^{N} c_n\, J_n(x),$$

where c_n, n = 0, 1, 2, · · · , N, are the unknown Bessel coefficients and N ≥ m. $J_n(x)$, n = 0, 1, 2,· · · , N are the Bessel polynomials of the first kind, which are defined as

$$J_n(x) = \sum_{k=0}^{\left[\frac{N-n}{2}\right]} \frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{(2k+n)}, \quad n \in N, \quad 0 \le x < \infty.$$

ISSN No:-2456-2165

1.1. Matrix Relations. First, we can write $J_n(x)$ in the matrix form, as follows:

$$\underbrace{\begin{pmatrix} J_0(x) \\ J_1(x) \\ J_2(x) \\ J_3(x) \\ \vdots \\ J_n(x_N) \end{pmatrix}}_{J_n^T(x)} = \underbrace{\begin{pmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{(\frac{N}{2})!(\frac{N}{2})!2^N} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N}{2})!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N+2}{2})!2^N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{(N-1)}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \\ x^N \end{pmatrix}}_{X^T(x)}$$

$$\mathbf{J}_n(x) = \mathbf{X}(x)\,\mathbf{D}^T, \qquad (2.1)$$

where

$$\mathbf{J}_n(x) = [J_0(x) \quad J_1(x) \cdots J_N(x)] \quad \text{and} \quad \mathbf{X}(x) = [1 \quad x \quad x^2 \quad x^3 \cdots x^N],$$

If N is odd,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!(\frac{N-1}{2})!2^{N-1}} & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{(\frac{N-1}{2})!(\frac{N+1}{2})!2^N} \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{(\frac{N-3}{2})!(\frac{N+1}{2})!2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{(N-1)}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{pmatrix}_{(N+1)\times(N+1)},$$

If N is even,

$$\mathbf{D} = \begin{pmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{(\frac{N}{2})!(\frac{N}{2})!2^N} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N}{2})!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{(\frac{N-2}{2})!(\frac{N+2}{2})!2^N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{(N-1)}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{pmatrix}_{(N+1)\times(N+1)}$$

Converting solution, y(x), to the matrix form. First, we consider the solution, y(x), of equation (1.1) defined by the truncated Bessel series given by equation (2). Then, the function is defined in the matrix form as

$$\begin{aligned} [y(x)] &= \mathbf{J}(x)\mathbf{C}, \\ &= \mathbf{X}(x)\,\mathbf{D}^T\,\mathbf{C}, \end{aligned} \qquad (2.2)$$

where

$$\mathbf{C} = [c_0\, c_1\, \cdots\, c_N]^T.$$

III. BESSEL METHOD OF SOLVING LINEAR SINGULAR HIGHER-ORDER BVP

This section describes the procedure for finding the numerical solutions of a linear singular higher-order BVP, namely, $v = 1$.

$$y^{(2r)}(x) + \frac{k_1}{x}y'(x) + \frac{k_2}{x^2}\,y(x) = g(x), \quad 0 \le x \le 1, \qquad (3.1)$$

There is singularity in the terms $\frac{k_1}{x}y'(x)$ and $\frac{k_2}{x^2}y(x)$ in equation (3.1). Thus, as x approaches zero in these terms, we employ L'Hopital's rule to eliminate the singularity at $x = 0$, as follows:

$$\lim_{x\to 0}\left[y^{(2r)}(x) + \frac{k_1}{x}y'(x) + \frac{k_2}{x^2}y(x)\right] = \lim_{x\to 0}[g(x)],$$

because

$$\lim_{x\to 0}\frac{k_1}{x}y'(x) = \frac{0}{0}, \quad \Rightarrow \quad \lim_{x\to 0}\frac{k_1}{1}y''(x) = k_1 y''(0),$$

Similarly,

$$\lim_{x\to 0}\frac{k_2}{x^2}y(x) = \frac{0}{0}, \quad \Rightarrow \quad \lim_{x\to 0}\frac{k_2}{2}y''(x) = \frac{k_2}{2}y''(0).$$

Based on the previous modulation, the BVP in equation (3.1) has the following form at point of singularity, x = 0:

$$y^{(2r)}(0) + \left(k_1 + \frac{k_2}{2}\right)y''(0) = g(0). \qquad (3.2)$$

The approximate solution, y(x), of equations (3.1) and (3.2) is

$$y(x) = \sum_{n=0}^{N} c_n\, J(x), \quad 0 \le x \le 1, \qquad (3.3)$$

where c_n, $0 \le n \le N$, are the unknown Bessel coefficients of $y(x)$ and N is a positive integer such that $2r \le N$. Then, the function can be written in the matrix form,

$$[y(x)] = \mathbf{X}\,\mathbf{D}^T\,\mathbf{C},$$

and its derivatives,

$$y^{(2r)}(x) = \sum_{n=0}^{N} c_n\, J(x), \quad r = 0, 1, 2, \ldots. \qquad (3.4)$$

The relation between matrix X(x) and its derivative, X (1), is

$$\begin{aligned} \mathbf{X}^{(1)}(x) &= \mathbf{X}(x)\mathbf{B}^T, \\ \mathbf{X}^{(2r)}(x) &= \mathbf{X}(x)\left(\mathbf{B}^T\right)^{2r}, \end{aligned} \qquad (3.5)$$

where

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

We have the following recurrence relations:

$$\begin{aligned} y^{(2r)}(x) &= \mathbf{X}^{(2r)}(x)\mathbf{D}^T\mathbf{C}, \\ &= \mathbf{X}(x)\left(\mathbf{B}^T\right)^{2r}\mathbf{D}^T\mathbf{C}, \quad r = 0,1,2,\cdots. \end{aligned} \tag{3.6}$$

We substitute the Bessel collocation points defined by

$$x_i = \frac{i}{N}, \qquad i = 0,1,\ldots,N,$$

into the approximate solution and its derivatives to realize the BVP in equations (3.1) and (3.2) at these points and acquire the following theorem: Theorem 3.1.

If the approximate solution of equations (3.1) and (3.2) is given by equation (3.3), the discrete Bessel system is given by

$$y^{(2r)}(x_i) + \frac{k_1}{x_i}y'(x_i) + \frac{k_2}{x_i^2}y(x_i) = g(x_i), \qquad 0 < x < 1, \tag{3.7}$$

The matrix pattern for this system is

$$\mathbf{W}\,\mathbf{C} = \mathbf{G}, \tag{3.8}$$

$$\mathbf{W} = \mathbf{X}\left(\mathbf{B}^T\right)^{2r}\mathbf{D}^T + \mathbf{P}_1\,\mathbf{X}\left(\mathbf{B}^T\right)\mathbf{D}^T + \mathbf{P}_2\,\mathbf{X}\,\mathbf{D}^T,$$

where

$$\mathbf{P_L} = \begin{bmatrix} \frac{K_l(x_0)}{x^l} & 0 & 0 & 0 \\ 0 & \frac{K_l(x_1)}{x^l} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{K_l(x_N)}{x^l} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}$$

Here, equation(3.8) corresponds to a system of (N + 1) algebraic equations with unknown Bessel coefficients c_0, c_1, $\cdots$, c_N . In addition, the matrix form, i.e., equation (3.2), for conditions can be written as

$$\mathbf{U}_j\mathbf{C} = [\lambda_j] \quad or \quad [\mathbf{U}_j\,;\lambda_j]\ ; \quad j = 0,1,2,\cdots,2r-1, \tag{3.9}$$

Where

$$\begin{aligned} \mathbf{U}_j &= \mathbf{X}(\mathbf{B}^T)^{2r}\mathbf{D}^T, \\ &= [u_{j0} \quad u_{j1} \quad u_{j2}\cdots \quad u_{jN}], \quad j = 0,1,2,\cdots,2r-1. \end{aligned}$$

We have the following new augmented matrix to obtain the solution of equation (3.1) under the conditions given by equation (3.2):

$$[\tilde{\mathbf{W}}\,;\tilde{\mathbf{G}}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & w_{32} & \cdots & w_{1N} & ; & g(x_1) \\ w_{20} & w_{21} & w_{32} & \cdots & w_{2N} & ; & g(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N-2r0} & w_{N-2r1} & w_{N-2r2} & \cdots & w_{N-2rN} & ; & g(x_{N-2r}) \\ u_{00} & u_{01} & u_{02} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1N} & ; & \lambda_1 \\ u_{20} & u_{21} & u_{22} & \cdots & u_{2N} & ; & \lambda_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{2r-10} & u_{2r-11} & u_{2r-12} & \cdots & u_{2r-1N} & ; & \lambda_{2r-1} \end{bmatrix}$$

IV. NONLINEAR HIGHER-ORDER BVP

This section describes the procedure for finding the numerical solutions of a nonlinear higher-order BVP, namely, $\nu > 1$. For simplicity, we consider $D = D^T$ and $B = B^T$ in this section. We require the following lemma:

Lemma 4.1. The following relation holds:

$$\begin{pmatrix} Y^\nu(x_0) \\ Y^\nu(x_1) \\ Y^\nu(x_2) \\ \vdots \\ Y^\nu(x_N) \end{pmatrix} = \begin{pmatrix} Y(x_0) & 0 & 0 & \ldots & 0 \\ 0 & Y(x_1) & 0 & \ldots & 0 \\ 0 & 0 & Y(x_2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & Y(x_N) \end{pmatrix}^{\nu-1} \begin{pmatrix} Y(x_0) \\ Y(x_1) \\ Y(x_2) \\ \vdots \\ Y(x_N) \end{pmatrix}$$

$$\begin{aligned} &= (\bar{\mathbf{Y}})^{\nu-1}\mathbf{Y} \\ &= (\bar{\mathbf{X}}\bar{\mathbf{D}}\bar{\mathbf{C}})^{\nu-1}\mathbf{XDC} \end{aligned}$$

We obtain the matrix representation.

Theorem 4.1. If the assumed approximate solution of equation (1.1) is equation (3.3), then the following system is obtained using the Bessel matrix method:

$$\mathbf{Q}\,\mathbf{C} = \mathbf{G}, \tag{4.1}$$

where

$$\mathbf{Q} = \mathbf{X}(\mathbf{B})^{2r}\mathbf{D} + \mathbf{P}_1\mathbf{XBD} + (\bar{\mathbf{X}}\bar{\mathbf{D}}\bar{\mathbf{C}})^{\nu-1}\mathbf{XD},$$

Replacing the boundary condition in equation (1.2), we have

$$\tilde{\mathbf{Q}}\mathbf{C} = \tilde{\mathbf{G}}, \tag{4.2}$$

Thus, we obtain a system which can be solved using nonlinear methods

V. NUMERICAL EXAMPLES

In this part, we present four examples to simulate the proposed algorithm to demonstrate accuracy and reliability of present method.

Example 5.1. [6, 8, 9, 10] Let us consider a nonlinear two-point BVP in astronomy.

$$u'' + \frac{2}{x}u' + u^5 = 0,$$

subject to the following boundary conditions:

$$u'(0) = 0, \quad \text{and} \quad u(1) = \sqrt{\frac{3}{4}}$$

The solution to this BVP is given by

$$u(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}.$$

Table 1 presents the maximum absolute error for the Bessel matrix method for different values of N. Table 2 compares the proposed method with other methods in terms of the maximum absolute error.

Table 1

N	Maximum absolute error
16	4.340E-11
18	1.780E-12
20	5.041E-14
23	1.250E-17

Table 2

Method	Maximum absolute error
Bessel matrix at N = 23	1.250E-17
Chebyshev polynomial[6]	3.330E-15
Method using Chebfun [6]	1.064E-15
Approximates using Tau [10]	6.670E-06
Collocation method [8]	1.170E-14
He's variational method [9]	3.920E-04

Example 5.2. [6] Consider the following equation:

$$z'' + \frac{2}{x}z' = -e^{-z} \qquad 0 < x \leq 1,$$

subject to the following boundary conditions:

$$z'(0) = 0,$$
$$0.1\, z(1) + z'(1) = 0,$$

The results are provided in the following Table 3.

Table 3

x	Proposed method	Chebyshev solution [6]	Chebfun solution[6]
0.0	1.147039019	1.14703914	1.14703902
0.1	1.146509642	1.14650977	1.14650964
0.2	1.144920502	1.14492063	1.14492050
0.3	1.142268564	1.14226869	1.14226856
0.4	1.138548748	1.13854887	1.13854875
0.5	1.133753903	1.13375402	1.13375390
0.6	1.127874756	1.13375402	1.13375390
0.7	1.120899861	1.12089998	1.12089986
0.8	1.112815520	1.12812640	1.11281552
0.9	1.103605704	1.10360582	1.10360570
1.0	1.093251945	1.09325206	1.09325195

Example 5.3. [1] Consider the following Higher order nonlinear BVP:

$$u^{(6)} + \frac{1}{x}u' + \frac{1}{x^2}u = 3 - e^{-x}u^2 - 4x + x^4e^{-x}(1-x)^2 \qquad 0 < x \leq 1,$$

subject to the following boundary conditions:

$$u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 2,$$
$$u(1) = 0, \quad u'(1) = -1, \quad u''(1) = -4.$$

The exact solution is given by

$$u(x) = x^2 - x^3.$$

Table 4 compares the numerical results obtained using the proposed method with the exact solution. Table 5 compares the results obtained using the proposed method and B-spline method [1].

Table 4

x	Exact solution	Bessel Matrix at $N = 7$
0.1	0.00900000	0.0090000436
0.2	0.03200000	0.0320002357
0.3	0.06300000	0.0630005109
0.4	0.09600000	0.0960007286
0.5	0.12500000	0.1250007835
0.6	0.14400000	0.1440006561
0.7	0.14700000	0.1470004135
0.8	0.12800000	0.1280001708
0.9	0.08100000	0.0810000281

Table 5

Bessel Matrix method at $N = 7$	B-Spline Method [1] at $N = 20$, $m = 2$
7.52671E-07	2.1700E-05

Example 5.4. [1] Let us consider the following linear higher order BVP:

$$u^{(8)} + \frac{1}{x}u' + \frac{1}{x^2}u = e^x\left[x^3 + 25x^2 + 172x + 336\right] \qquad 0 < x \leq 1,$$

subject to the following boundary conditions:

$$u(0) = u'(0) = u''(0) = 0, \quad u'''(0) = 6,$$
$$u(1) = e, \quad u' = 4\,e, \quad u''(1) = 13\,e, \quad u'''(1) = 34\,e.$$

The exact solution is given by

ISSN No:-2456-2165

$$u(x) = x^3 e^x.$$

Table 6 compares the results obtained using the proposed method and the exact solution

Table 6

x	Exact solution	Bessel matrix method at $N = 20$
0.1	0.00110517091807564	0.00110517091807565
0.2	0.00977122206528135	0.00977122206528141
0.3	0.03644618780455208	0.03644618780455224
0.4	0.09547678064904130	0.09547678064904161
0.5	0.20609015883751601	0.20609015883751645
0.6	0.39357766088434993	0.39357766088435039
0.7	0.69071717866237344	0.69071717866237379
0.8	1.13947695538814341	1.13947695538814363
0.9	1.79305066803341630	1.79305066803341647

Table 7 compares the proposed method and B-spline method in terms of the maximum absolute error.

Table 7

Bessel matrix method at $N = 20$	B-spline method [1] at $N = 20$
4.49E-16	2.9801E-08

VI. CONCLUSION

This study shows how Bessel Matrix method can be applied for computing the solution of linear and nonlinear BVP. This technique is simple to implement, effective and yields accurate results. Illustrative examples have been provided and comparisons are made with existing methods.

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