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Statistical Solution to Complex Markov Matrix Chains

Dr. Ismail Abbas, Senior Lecturer at MTC, Cairo University

Abstract:- We divide this article into two parts. In part 1, we introduce a complex Markov matrix, propose and validate a statistical technique to its solution called the complex stationary eigenvector of Markov chains. We show that statistical techniques are more efficient and more precise than the classical algebraic method of solving a linear system of algebraic equations of the homogeneous Markov system. The statistical solution fails only when the Markov matrix is not invertible. In this case, the classic solution also fails.

In part 2, we introduce a stochastic transition matrix B other than the Markov transition matrix. The transition matrix B can be real or complex as well as the Markov matrix. Likewise, we propose and validate a statistical solution to complex B-Matrix transition chains.nThe proposed B-Matrix (nxn) and its B-Matrix chains is valid for any 2D and 3D configuration for any arbitrary number of free nodes n. In addition, we extend the validity of the hypothesis principle applied for real B-Matrix chains to the case of complex B-Matrix chains: [For a positive symmetric physical matrix, the sum of their powers at the eigenvalues is equal to the eigenvalue of their sum of the series of powers of the matrix]. In the current article, we provide a numerical validation of this principle by comparing the eigenvalue of the sum of the series of powers of the matrix B with the sum of the series of infinite powers of the eigenvalues of the B-matrix itself.

I. INTRODUCTION

By complex Markov matrix we mean that one or more of its inputs are complex or contain imaginary numbers.

A complex number is a number which can be expressed as X + i Y, where X and Y are real numbers, and i is called the imaginary unit, and satisfying the equation i ^ 2 = -1.

Although no physical quantity is complex in nature, complex numbers have the advantage of describing the physical quantity by two real numbers, for example one for the amplitude and the other for the phase.The most common example is applied to linear pulse vector and alternating current (AC) circuits to describe complex potential V and complex current I as well as their complex product for active and reactive power ... etc.

In addition, complex numbers are convenient for the mathematical description of waves. This is because of Euler's famous formula: Ae ^ iϕ = Acosϕ + iAsinϕ. with the real quantity A serving as amplitude and the also real number ϕ in radians, its phase.In a way, complex numbers and complex matrices are the extension of real numbers and real matrices.

We have discussed in previous articles real Markov matrix chains and real B matrix chains [2,4]. In the current article, we answer the question, what if the Markov transition matrix or the transition B-matrix is complex, which is the subject of this work. For this target, this paper presents a complex Markov matrix Ac as well as a complex transition matrix Bc and examines the stochastic behavior for both.

II. THEORY

For convenience, we divide this article into two parts, part 1 for complex Markov chains and part 2 for complex B matrix chains.

PART 1

Complex Markov matrix chains

The real Markov transition matrix A (nxn) is defined by two conditions i and ii, [5]

i-All its entries a i, j are elements of the closed interval [0,1]

ii- The sum of the entries for all rows / or all columns is equal to 1.

We add a complementary condition iii,

iii-The matrix A must be invertible or not singular.

Condition iii is not present in the original definition of the stochastic Markov transition matrix but it seems right and important to add it as the third condition rarely discussed before [4].

Regarding condition iii, if the Markov matrix A is not invertible, the steady-state solution of the Markov chain could diverge or converge to erroneous results.[4]

Now consider the case where the Markov transition matrix is complex:

Conditions i, ii and iii should also be fulfilled in one way or another.

However, for condition i, it is obvious that a complex aij is not an element of [0,1] but a possible compensation is that the norm [$a_{i,j}$] or Sqrt (X ^ 2 + Y ^ 2) should fall in the interval [0,1].

To be precise and not to go into much detail, here we present an arbitrarily chosen complex Markov matrix Ac (6X6),

Ac =
{.4, 0, .2-.3i, .1, .2 + .3i, .1},
{.3, .2, .2, 0, .3, 0},
{0, .6, .1, .1, .2, 0},
{.1, .3, 0, .3, .1, .2},
{0, .1, .2, .3, 0, .4},
{.2, 0, .1, 0, .2, .5}

Note that Ac satisfies conditions i, ii and iii.

We can apply the statistical solution described in [2,4] to find the complex eigenvector Vc called the stationary probability solution of the Markov transition matrix which satisfies the equality,

Vc=Vc . Ac

where Vc is given by,

Vc = Ac ^ N.. . . . . (1)

For N a sufficiently large number.

We used a simple double precision algorithm to calculate Vc using Equation 1 and the resulting steady state probability vector for Ac or its complex eigenvector Vc = X + iY was found numerically as follows:

Vc =
.1787-.0020138 i,
.17074-.031571 i,
.13959-.053815 i,
.11896 + .0096896 i,
.17148 + .041237 i
.22051 + .036466 i

It is simple to verify that Vc is the precise complex eigenvector of the complex matrix Ac represented above and is associated with the real eigenvalue of the unit.

In other words, we have checked the equation,

Vc. Ac = Vc. . . . . (2)

and found it to be accurate to 5 decimal places.

It is important to note here that the above Markov matrix is an RHS matrix and not an LHS matrix.

It is also important to note that the sum of the real parts of X in Vc above is 0.99998 (close to unity) and the imaginary parts of Y add to -8E- (6) (close to zero) which shows:

i-The high precision of the statistical method proposed to solve complex Markov chains.

ii- Complex Markov transition matrix chains are useful as a conservative solution in a closed system for the temporal evolution of a complex initial state vector.

The conclusion is that a real Markov matrix A is practical to describe the temporal evolution of scalar quantities such as the number of particles or objects or the energy density field in real space R and the complex matrix of Markov Ac is also useful to describe the temporal evolution of a set of complex numbers or quantities expressed both in amplitude and in phase.

PART 2

B-Matrix chains

The real and / or complex transition matrix B is different in inputs and digital processing from the real and / or complex Markov transition matrix A and both operate under different specific statistical and physical conditions.

The formulation and processing of the complex form of the transition matrix B and its time chains are analogous to those followed for the real matrix B explained in [1,2,3] except that the probability rate coefficients b_{ij} are complex. The digital spatio-temporal diffusion equation for the energy density U (x, y, z, t) in matrix form is given by the recurrence relation of B-Matrix,

$U^{N+1}_{i,j,k,}$ = B. $(b + S)^{N}_{i,j,k}$ + B ^ N .U (0). . . . . (3)

b is the vector of the boundary conditions, S is the term of the source / sink vector and U (0) is the initial conditions.

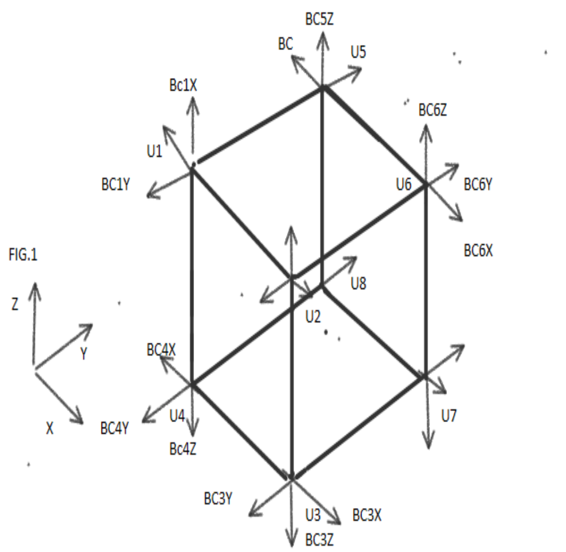
It is obvious that b and S can be complex numbers in this case.

It has been shown [1, 2, 3] that the interactions of b and S with the adjacent free nodes are the same as the interaction between two adjacent free nodes (bi, j = 1/6 for the 3D configuration) as on Fig 1 and Fig.2.

Here is a correct formulation of the complex matrix Bc (8X8) corresponding to the free nodes of figure 1, which is the simplest 3D geometry. It shows the geometric configuration and the geometry of the boundaries.

Note that the unconventional statistical method used to numerically solve the heat diffusion equation and the Laplace and Poisson equations is more stable and accurate than the Mat Lap or the conventional finite difference FDM methods widely used in the numerical solution of such equations with partial derivatives [9]

Fig.1. 8 free nodes and 24 Dirichlet Boundary conditions (reduced to 8 BC)



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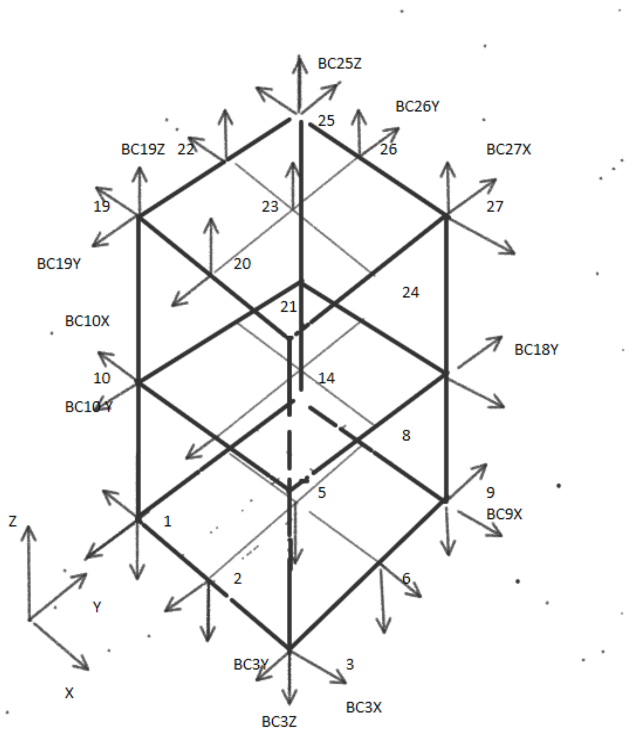

Fig.2.27 free nodes and 54 Dirichlet boundary conditions (reduced to 27)

With regard to figure 1, we describe a direct statistical method to find the steady-state solution for U (X) when the time t = Ndt tends to a large infinite number in three precise consecutive steps similar to that followed for the matrix B real.

Step 1
Formulate the complex matrix Bc (8X8) and the vector of Dirichlet boundary conditions vector b arranged in the correct order.
The complex transition matrix Bc of figure 1 is given in a similar way to the real transition matrix B [1,3] but replacing 1 by the imaginary unit i as follows,

Bc =
{RO, i / 6-RO / 6, i / 6-RO / 6, 0, i / 6-RO / 6, 0, 0, 0},
{i / 6-RO / 6, RO, 0, i / 6-RO / 6, 0, i / 6-RO / 6, 0, 0},
{i / 6-RO / 6, 0, RO, i / 6-RO / 6, 0, 0, i / 6-RO / 6, 0},
{0, i / 6-RO / 6, i / 6-RO / 6, RO, 0, 0, 0, i / 6-RO / 6},
{i / 6-RO / 6, 0, 0, 0, RO, i / 6-RO / 6, i / 6-RO / 6, 0},
{0, i / 6-RO / 6, 0, 0, i / 6-RO / 6, RO, 0, i / 6-RO / 6},
{0, 0, i / 6-RO / 6, 0, i / 6-RO / 6, 0, RO, i / 6-RO / 6},
{0, 0, 0, i / 6-RO / 6, 0, i / 6-RO / 6, i / 6-RO / 6, RO}

It is proposed that the main diagonal entry RO be constant over the entire main diagonal and that it has special statistical physical significance .

Consider the complex transition matrix Bc for RO = 0 is given by,
Bc =
{0, i / 6, i / 6, 0, i / 6, 0, 0, 0},
{i / 6, 0, 0, i / 6, 0, i / 6, 0, 0},
{i / 6, 0, 0, i / 6, 0, 0, i / 6, 0},
{0, i / 6, i / 6, 0, 0, 0, 0, i / 6},
{i / 6, 0, 0, 0, 0, i / 6, i / 6, 0},
{0, i / 6, 0, 0, i / 6, 0, 0, i / 6},
{0, 0, i / 6, 0, i / 6, 0, 0, i / 6},
{0, 0, 0, i / 6, 0, i / 6, i / 6, 0}

It is clear that the complex eigenvalue of Bc for RO = 0 , is
evBc = i / 2 . . . . (4)
*step 2*Find the matrix I-Bc, where I is the unit matrix, hence I-Bc is expressed below by,
I-Bc=
{1, -i / 6, -i / 6, 0, -i / 6, 0, 0, 0},
{-i / 6, 1, 0, -i / 6, 0, -i / 6, 0, 0},
{-i / 6, 0, 1, -i / 6, 0, 0, -i / 6, 0},
{0, -i / 6, -i / 6, 1, 0, 0, 0, -i / 6},
{-i / 6, 0, 0, 0, 1, -i / 6, -i / 6, 0},
{0, -i / 6, 0, 0, -i / 6, 1, 0, -i / 6},
{0, 0, -i / 6, 0, -i / 6, 0, 1, -i / 6},
{0, 0, 0, -i / 6, 0, -i / 6, -i / 6, 1}

Step 3
Find the complex transfer matrix Ec by inverting the I-Bc matrix which is guaranteed to be non-singular and unique,
Ec = (I-Bc) ^ - 1 .. . . (5)

The origin of the matrix Ec is in fact the power series of the transition matrix Bc, i.e.
Ec = Bc ^ 0 + Bc ^ 2 + Bc ^ 3 +. . . . . + Bc ^ N.. . . . . (6)
With Bc ^ 0 = I

In other words, Ec satisfies both equations 5 and 6.Now it is simple to find the Eigenvlue for Ec like
evEc = .8 + .4i. . . . . (7),

Therefore, we can advance, to find the complex steady-state transfer matrix Dc given by the equality,
Dc = Ec - I.. . . . . . (8)

Therefore, the complex transfer matrix Dc is given by,
Dc=
{-.07027, .14054i, .14054i, -.043243, .14054i, -.043243, -.043243, -.021622i},
{.14054i, -.07027, -.043243, .14054i, -.043243, .14054i, -.021622i, -.043243},
{.14054i, -.043243, -.07027, .14054i, -.043243, -.021622i, .14054i, -.043243},
{-.043243, .14054i, .14054i, -.07027, -.021622i, -.043243, -.043243, .14054i},
{.14054i, -.043243, -.043243, -.021622i, -.07027, .14054i, .14054i, -.043243},
{-.043243, .14054i, -.021622i, -.043243, .14054i, -.07027, -.043243, .14054i},
{-.043243, -.021622i, .14054i, -.043243, .14054i, -.043243, -.07027, .14054i},
{-.021622i, -.043243, -.043243, .14054i, -.043243, .14054i, .14054i, -.07027}

We can easily show that the complex eigenvalue of Dc, i.e. evDc calculated from the matrix above, is given by,
evDc = -0.2 + 0.4 i. . . . (9)

The importance of the matrix Dc is that it gives the steady-state equilibrium solution vector to the complex potential field U (x) for sufficiently large number of jumps or time steps N>>1.

$U^{N+1}_{i,j,k}$ = Dc. $(b + S)^{N}_{i,j,k}$ + Bc ^ (N + 1) .U (0). . . . . (10)

Equations 5, 6 and 10 are valid for real and complex B-Matrix chains.

b is the vector of the boundary conditions arranged in the correct order, S is the source / sink complex term and U (0) is the initial conditions.

It has been shown [1,2,4] that the interactions or probability rate coefficients of BC and S with the adjacent free nodes are the same as the interaction between two adjacent free nodes bi, j (bi, j = 1 / 6 for 3D configuration) as shown in Fig. 1 and Fig. 2.

III. VALIDATION OF NUMERICAL RESULTS

The validation of the numerical results is carried out first of all for the complex eigenvalue of the matrix Dc (evDc)then for the solution of the complex vector U (x) by applying the matrix Dc defined by Eq. 8.

III-A. Validation of the eigenvalue evDc

The complex eigenvalue of the matrix Dc turned out to be,

evDc = -0.20000 + .0.40000 i. . . . . (9)

The matrix Bc being a statistical physical matrix, it follows that:

If the eigenvalue of the matrix Bc is ev1 and the eigenvalue of the matrix Bc ^ 2 is ev2, that of the matrix Bc ^ 3 is ev3. . . etc,

then, we have: [1,2,3]

ev2 = ev1 ^ 2

ev3 = ev ^ 3.

. . ..

evN = ev ^ N ....

Note that when N approaches an infinitely large number, evN approaches zero, which is a necessary condition for the convergence of the Ec and Dc matrices in equations 5 and 10.

The hypothesis explained in the previous articles states that :, [1,2,3]

[For a positive symmetric physical matrix, the sum of their powers at the eigenvalues is equal to the eigenvalue of their sum of the matrix power series]. . . . . . Principle 1.

Principle 1 is valid for the real transition matrix B as well as for the complex transition matrix B.

In other words, if the matrix Dc is expressed as a power series of the transition matrix Bc,

Dc=Bc+Bc^2+Bc^3 + ... +Bc^N

then the eigenvalue of Dc can also be expressed as a power series of the eigenvalues of Bc in a similar way, i.e.

evDc=evBc+evBc^2 +evBc^3 +. . . . .+evBc ^N

For N sufficiently large. “ Principal 1”

In other words,The mathematical formulation of principle 1 consists of the following two equations,

Dc = Bc + Bc ^ 2 + Bc ^ 3. ... + Bc ^ N.. . . . . (11)

Then,

evD = evB + evB ^ 2 + evB ^ 3 +. . . . . . . + evB ^ N.. . (12)

Equations 11 and 12 are valid for N sufficiently large number.

In Eq. 11 and 12, evBc and evDc are replaced by evB and evD so as not to overload the equation.Therefore, using equations (4 and 12), we can find an alternative numerical calculation for evD as,

evD = i / 2 + (i / 2) ^ 2 +. . . . . + (i / 2) ^ N.. . . . . . (11)

As N goes to infinite large number.

Using a simple, double-precision algorithm, we evaluated the power series sum of equation (11) as follows:

evD = -0.20000 + 0.40000 i. . . . . . (12)

Which exactly is the numerical result of equation 9. This means that if we inspect the digital result of Eq. 12, we find it in perfect agreement with Eq. 9 which numerically validates the correctness of the chains of the complex matrix Bc and of the chains of power series Dc and also validates Principle 1 for complex stochastic transition chains.

B-Validation by vector solution

TThe steady-state solution of Eq. 10 for zero initial conditions, (S (0) = 0) will be column vectors with complex inputs given by:

Uc = b. Dc. . . . . (13)

It is simple to find the vector of boundary conditions b, real or complex, and to multiply it by the complex matrix Dc above.

The following numerical results are obtained for certain arbitrary boundary conditions chosen.

a) = {1, 1, 1, 1, 1, 1, 1, 1}

Uc = (- 0.20000 + 0.40000 * i, -0.20000 + 0.40000 * i, -0.20000 + 0.40000 * i, -0.20000 + 0.40000 * i, -0.20000 + 0.40000 * i, -0.20000 + 0.40000 * i, -0.20000 + 0.40000 * i, -0.20000 + 0.40000 * i}

Here, in case a), we find an interesting numerical result. Although all of the system inputs, BC and / or S are real, the resulting voltage distribution is complex, i.e. if the Input voltage on BC or S = 1 cos wt

The voltage distribution for all free nodes would be -02 cos wt + 0.4 sin wt

the imaginary unit i changes the function cos wt to sin one.

b) b = {i, i, i, i, i, i, i, i}

Uc = ({{- 0.40000-0.20000 * i, -0.40000-0.20000 * i, -0.40000-0.20000 * i, -0.40000-0.20000 * i, -0.40000-

0.20000 * i, -0.40000-0.20000 * i, -0.40000- 0.0000 * i, -0.40000-020000 * i}
c) b = {1, 0, 0, 0, 0, 0, 0, 0}
Uc =
(-0.07027, 0.14054 * i, 0.14054 * i, -0.043243, 0.14054 * i, -0.043243, -0.043243, -0.021622 * i}
d) b is identical to zero but S = 2 in voltage units placed at the cube corner free node point 4
S = {0, 0, 0, 2, 0, 0, 0, 0}
Uc = {- 0.086486, 0.28108 * i, 0.28108 * i, -0.14054, -0.043244 * i, -0.086486, -0.086486, 0.28108 * i}
and so on for any real or complex boundary condition chosen arbitrarily and for the source term S.
Note that BC and S are treated the same.

IV. CONCLUSION

In part 1, it is shown that the proposed statistics

The solution for complex Markov chains is fast, stable and precise.

If the Markov matrix is singular or not invertible, there is no stable convergence towards the equilibrium state and the new method fails as well as the other classical methods based on the resolution of a homogeneous system of algebraic equations.

In part 2, the application of the hypothesis known as principle 1 used in the previous articles [1,2,3] leads to a new formula for series of infinite powers,

namely the sum [(i / 2) ^ N] is equal to -0.2 + 0.4 i, which numerically proves the correctness of the hypothesis itself. It validates in fact the proposed principle: [For positive symmetrical physical matrix, the sum of their eigenvalues powers is equal to the eigenvalue of their sum of the series of the powers of the matrix].

In short, Markov matrix chains and B transition matrix chains can be complex and the proposed statistics work with precision and efficiency. The complex transfer matrix Dc yields correct solution for the complex 3D voltage domain.

N.B. All calculations in this article have been produced with the author's double precision algorithm to ensure maximum precision, as followed by Ref. 8 for example

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