

THEORY OF B-MATRIX AND HYPERCUBE

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Abstract:- We introduce and define a theory of the B-stochastic transition matrix other than the Markov stochastic transition matrix. We also define and explain the main assumptions and principles essential to its validity as well as its inherent characteristics. We compare the characteristics of matrices B and Markov matrix and show that both matrices can be real or imaginary and that their chains work in both real and imaginary spaces. In particular, matrix B has a striking advantage of being easy to formulate and simple to manage for 2D and 3D spatiotemporal diffusion problems with any arbitrary boundary conditions BC and any arbitrary configuration of source / sink terms such as case of Poisson and Laplace partial differential equations as well as heat diffusion PDE. Finally, we propose a modeling of the 16 vertices of the 4D hypercube in the cartesian space x, y, z, w by a super symmetrical matrix B of 16 inputs and 16 outputs.

I. INTRODUCTION

We have introduced and defined a B-stochastic transition matrix (2,4) other than the Markov stochastic transition matrix M ..

The aim of this article is to further define and explain the theory and applications of the stochastic transition matrix B and to compare its main characteristics with the Markov matrix.

Matrix B has a striking advantage that it is easy to formulate and simple to handle for a 2D and 3D spatiotemporal scattering problem with arbitrary boundary conditions BC and an arbitrary configuration of the source / sink terms. In fact, the formulation of the matrix B follows from its geometry and the solution of the chains of the transition matrix B comes from the addition of the terms of its series of powers.

In order to further study the theory and properties of the proposed matrix B and its chains, it is better to first present a brief investigation of the well-known Markov transition matrix chains.

Originally, the real Markov transition matrix $M_{i,j}$ ($n \times n$) where $M_{i,j}$ are elements of R is defined by the following conditions i and ii, [1]

- i-All its inputs $M_{i,j}$ are elements of the closed interval $[0,1]$
- ii- The sum of the entries $M_{i,j}$ for all the rows / or all the columns is equal to 1. In ref [2], we have proposed another

essential condition absent or complementary condition iii as follows,

- iii-The Markov matrix M must be invertible or non-singular.

Condition iii is absent from the original definition of the stochastic Markov transition matrix but it seems right and important to include it.

Although The third condition iii is extremely important, it rarely discussed in the present literature. [2,3].

Concerning condition iii, if the Markov matrix A is not invertible, then it is impossible to handle it correctly in a classical or statistical manner. The Markov solution or steady-state chain called the stationary probability eigenvector could diverge or converge towards erroneous results [2]. In other words, the unitary stationary probability eigenvector does not exist.

Now consider the case where the Markov transition matrix is complex, i.e. one or more inputs of $M_{i,j}$ is complex, [2], then the conditions i, ii and iii must also be fulfilled in one way or another. However, for condition i, it is obvious that a complex $M_{ij} = X + iY$, is not an element of $[0,1]$ but a possible compensation is that the norm $[M_{i,j}]$ or $\sqrt{X^2 + Y^2}$, must lie in the interval $[0,1]$.

Moreover, one can easily show that if M is a Markov matrix of principal eigenvalue 1, then iM is also a Markov matrix of principal eigenvalue i where i is the imaginary unit $\sqrt{-1}$

However, the stochastic Markov matrix mainly concerns the treatment of the temporal evolution of a set of initial values in a closed system subjected to the conditions of transfer of pairs of probabilities of the inputs of M-Matrix which is $M_{i,j}$.

It has the defect of not being able to take into account the boundary conditions of the system BC or its source / sink term S .

This defect is one of the reasons for the introduction of the B-stochastic transition matrix theory and its applications which have been shown to be able to deal with many physical situations, including boundary conditions BC and source / sink terms S such as those present in Laplace and Poisson PDEs as well as the heat diffusion equation [3,4].

This is a consequence of the fact that the stochastic transition matrix B can replace the FUNCTIONAL DESCRIPTION of the boundary value problem..Itcan replace the spatio-temporal PDE with arbitrary BC and IC . The required digital description of the spatio-temporal evolution of the PDE solution is found directly from this matrix representation.

For this objective, we first define the B-stochastic transition matrix through four rigorous statistical assumptions [4] and two essential hypothesis. Hereafter, the 4 assumptions or statistical conditions that the inputs of the statistical transition matrix $(B_{i,j})$ must satisfy in the configuration space of the Cartesian coordinates 2D and 3D:

- i- The sum of $B_{i,j} = 1$ for all the rows far from the borders and the sum $B_{i,j} < 1$ for all the rows adjacent to the borders meaning that the probability of the whole space = 1.
- ii- $B_{i,j} = 1/4$ for i adjacent to j .. and $B_{i,j} = 0$ otherwise in a 2D geometry while $B_{i,j} = 1/6$ for i adjacent to j in 3D geometry.

Condition ii formulates the principle of an equal a priori probability and that the probability of the whole sampling space = 1.

- iii- $B_{i,i} = RO$, i.e. the main diagonal is made up of constant inputs RO .

Obviously $0 \leq RO \leq 1$.

RO has great importance and is most significant in describing the transition matrix B and its applications.

For example, in the heat diffusion equation, RO can take any value in the closed interval $[0,1]$ depending on the coefficient of thermal conductivity a [3], while for Laplace and Poisson PDE, $RO = 0$.

That is to say that B is a null principal diagonal matrix in the problems of electrostatic tension which corresponds to the assumption of a null residue after each time step dt for all the free nodes.

- .iv- $B_{i,j} = B_{j,i}$, for all i, j .

The matrix B is essentially symmetrical to conform to the physical principle of detailed balance and the symmetry of laws of nature itself.

Obviously, the stochastic matrix B is very different from the Laplacian mathematical transfer matrix A explained in numerous articles on numerical methods for solving a linear system of algebraic equations [6] and also different from the stochastic transition matrix of Markov M .

The statistical and physical significance of the nature of B is clear from conditions i-iv, as it is not a pure mathematical transition of mathematical variables or states such as Markov but rather a transition in space and time (x, t) of physical

quantities such as scalar energy density in heat and voltage problems.

II. THEORY

Actually, matrix B by itself does not support the solution by itself, but it is the core of the matrix solution and must be processed to produce the transfer matrices E and D which are the solution as explained in the following section.

The partial differential heat diffusion equation and Poisson's PDE can be formulated as follows:

$$dU / dt \text{ partial} = a \text{ Nabla}^2 U + S \dots \dots \dots (1)$$

subject to the boundary conditions of Dirichlet or Neumann BC and to the initial conditions IC.

U is the thermal energy density or voltage potential energy expressed in J / m^3

S is the energy density source/sink term..

In the solution of the statistical matrix B chains, Eq.1 is deleted and the spatiotemporal functional description of U is replaced by the description of the transition matrix B and its power series.

The theory of the B-stochastic transition matrix contains two main Hypothesis :

I: For each 2D and 3D geometrical configuration, there exists a stochastic transition matrix B such that the following recurrence relation is valid,

$$U^{N+1}(x, t + dt) = B \cdot (U^N + b + S) + B^N \cdot U(x, 0) \dots \dots \dots (2)$$

where N is the number of iterations.

b is the vector BC arranged in the proper order.

$U(x, 0)$ are the initial conditions IC.

Hypothesis II:

The eigenvalue of the symmetric matrix B called evB satisfies the following relations,

$$(evB)^2 = ev(B^2)$$

$$(evB)^3 = ev(B^3)$$

.....

$$(evB)^N = ev(B^N) \dots \dots \dots (3)$$

where $ev(B^N)$ is the eigenvalue of the matrix B raised to the power N .

Moreover, Hypthesis I and II suggest Principle I stated as follows:

[For a positive symmetric stochastic matrix, the sum of their eigenvalue powers is equal to the eigenvalue of their sum of the series of powers of the matrix] Principle I.

Principle 1 suggests the definition of the transfer matrix of power series E and D as follows:

$$E(N) = B^0 + B + B^2 + \dots + B^N \dots \dots \dots (4)$$

Equation 4 define the statistical transfer matrix E by the series of powers of the matrix B ,

with $B^0 = I$,
 And the matrix D is defined as,
 $D = E - I \dots \dots \dots (5)$

In other words, the transition matrix B defined by the conditions i-iv is the kernel or the starting point to obtain the time-dependent transitory solution of the diffusion problem, for the finite value of N at $t = Ndt$ or the steady state equilibrium solution when N goes to sufficiently large values.

In all cases, the matrix solution of Eq. 1, that is, the search for B, D and E, for a given geometry, must be processed in 3 precise consecutive steps to produce the transfer destination matrix D as follows:

1- First, we discretize the 2D or 3D domain into n equidistant free nodes, and find the appropriate stochastic transition matrix B (nxn) through using the conditions i-iv above.

The solution follows from the successive application of the recurrence formula: $U^{(N+1)} = B (U^N + b + S)$ where $N = 0, 1, 2, \dots, N$.

That is to say that the solution U (x, t) at iteration N or at time = N dt is given by:

$$U^N = (B^0 + B + B^2 + \dots + B^N) \cdot (b + S) + B^N \cdot U(0, x) \dots \dots (6)$$

Expressed in power series of matrix B.

$U(x, 0)$ is the initial conditions IC and converges to zero for large N when B^N itself converges to zero.

Obviously, all the entries of the matrix B^N converge towards zero when N tends towards an infinitely large number which is a necessary condition for the convergence of the matrix E itself. At the limit where N tends to an infinitely large number, the term $IC = U(x, 0)$. B^N disappears and the matrix E and D become stationary.

We seek to arrive at a simple formula for the required steady state solution described by Eq. 4, i.e.:

$$U(N) = D(N) (b + S) \dots \dots \dots (7)$$

b is the vector of the boundary conditions arranged in the proper order and S is the source / sink vector located at the free nodes. $S = 0$ in LPDE.

$U(N)(x, t)$ is the spatiotemporal solution vector of the situation described by Poisson PDE (1). N represents the number of jumping steps in time dt or the number of iterations N.

2- In a second step, we define b which is the vector of the boundary conditions by arranging BC in the right order and calculate the source / sink term vector in energy density J / m^3 rather than the voltage in volts or the temperature in degrees Kelvin (for the case of the heat diffusion equation).

Note that BC and S are treated the same in the B-Stochastic transition matrix chain procedure.

3-In a third step, calculate the transfer matrix E and the destination transfer matrix D as follows:

$$E(N) = B^0 + B + B^2 + \dots + B^N \dots \dots \dots (8)$$

and the destination or solution matrix D as,

$$D(N) = E(N) - I \dots \dots \dots (9)$$

In fact, it is not complicated to calculate the matrix E from equation 4 by adding the power series for a finite number of time steps N and the solution U (x, t) will be the transient or time dependent PDE solution[2,3] . Alternatively ,E can be calculated by using equality,

$$E = (I - B)^{-1} \dots \dots \dots (10)$$

which is valid for sufficiently large values of N and the solution U (x) represents the steady-state equilibrium solution. The simplicity and precision of the proposed numerical statistical method of matrix B are quite striking,

There is no need to mathematically manipulate the matrix or use a known MATLAB to find the solution as it is inherent in the B-Matrix itself, which contains all the required information.

It suffices first to build the matrix B from where the transfer matrix E and D are found. The boundary condition vector b is calculated according to the geometric configuration of the problem and the source / sink term corresponding to the free nodes is calculated according to its actual values and placed correctly in its spatial nodes. then finally to calculate the matrix solution by Eq. (7)

In order not to worry too much about the details of the theory, let's go right into 2D and 3D illustrative applications.

Applications are divided : Application A includes imaginary B-chains, Application B presents cases of symmetry and supersymmetry of matrices B and D, and Application C presents a prediction model for the hypercube.

III. APPLICATIONS

III-A . IMAGINARY B-MATRIX

The use of the imaginary word to describe the coefficients of the matrix is rather unfortunate. This gives the impression that imaginary numbers are sort of fictitious entities. But they are,in many cases, as "real" as any other type of number.

The same goes for the stochastic Markov transition matrix where the conditions i, ii can be replaced by,

- i-All its inputs M_{ij} are elements of the closed interval $[0,1]$
- ii- The sum of the entries M_{ij} for all the rows / or all the columns is equal to 1.

In a previous article [2], we discussed the possibility of extending the chains of matrix B to imaginary space where the whole probability space is assumed equal to 1 instead of 1 in real space and we obtained the result for the eigenvalue of the complex matrix D as $ev_{DC} = 0.2 + 0.4i$ for $RO = 0$.

Here we extend the calculations to different imaginary values of RO in the interval $[0, 1]$

That is $0, 0.2i, 0.4i$ etc .

As an example of calculation, consider the case of the 3D cube of 8 free nodes in figure 4 when $RO = 0.4i$

We calculate the complex matrices $B(8 \times 8)$ and $D(8 \times 8)$ as follows,

$$Bc = \{0.4i, i/6-0.4i/6, 0, i/6-0.4i/6, i/6-0.4i/6, 0, 0, 0\}, \\ \{i/6-0.4i/6, 0.4i, i/6-0.4i/6, 0, 0, i/6-0.4i/6, 0, 0\}, \\ \{0, i/6-0.4i/6, 0.4i, i/6-0.4i/6, 0, 0, i/6-0.4i/6, 0\}, \\ \{i/6-0.4i/6, 0, i/6-0.4i/6, 0.4i, 0, 0, 0, i/6-0.4i/6\}, \\ \{i/6-0.4i/6, 0, 0, 0, 0.4i, i/6-0.4i/6, 0, i/6-0.4i/6\}, \\ \{0, i/6-0.4i/6, 0, 0, i/6-0.4i/6, 0.4i, i/6-0.4i/6, 0\}, \\ \{0, 0, i/6-0.4i/6, 0, 0, i/6-0.4i/6, 0.4i, i/6-0.4i/6\}, \\ \{0, 0, 0, i/6-0.4i/6, i/6-0.4i/6, 0, i/6-0.4i/6, 0.4i\}$$

The eigenvalue of the complex matrix B above is ,,

$$ev_{Bc} = 0.7i \dots \dots (11)$$

And the complex transfer matrix $E_c = (I - Bc)^{-1}$, is given by,

$$E_c = \{0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, 0.0041669-0.00044040 * i, -0.0070239-0.013303 * i\}, \\ \{-0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i\}, \\ \{0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, 0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i\}, \\ \{-0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i\}, \\ \{-0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i, -0.0070239-0.013303 * i\}, \\ \{0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, 0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i\}, \\ \{-0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i\}$$

$$0, 0.13303 * i, 0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i\}, \\ \{-0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, 0, 0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, 0.85169 + 0, 32431 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0, 0.054549 + 0.061945 * i\}, \\ \{-0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, 0.0041669-0, 0.0044040 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i\}, \\ \{0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, -0.0070239-0.013303 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i, -0.054549 + 0.061945 * i\}, \\ \{-0.0070239-0.013303 * i, 0, 0.0041669-0.00044040 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, -0.054549 + 0, 0.061945 * i, -0.0070239-0.013303 * i, -0.054549 + 0.061945 * i, 0.85169 + 0.32431 * i\}$$

Clearly, the transfer destination matrix D_c is equal to $E-I$ and has a complex eigenvector ev_{Dc} equal to i and its eigenvector V_c is given by, $\{0.67114 + 0.46980 * i\}$,

In exact agreement with the sum of the powers that is:

$$0.7i + (0.7i)^2 + (0.7i)^3 + \dots + (0.7i)^N = ev_{Dc} = 0.67114 + 0.46980i, \text{ for } N \text{ sufficiently large.}$$

The imaginary admitted values of RO are between 0 and 1 and we can find the corresponding eigenvalue of the complex destination matrix D (ev_{Dc}) and compare with the power summation of equations 8 and 9 which is alternately equal to the formula,

$$ev_{Dc} = 1 / (1 - ev_{Bc}) - 1 \dots \dots 12$$

When we continue we get the results for the complex eigenvalues of BC (ev_{Bc}), and the corresponding eigenvalues of DC (ev_{Dc}). The results are presented in Table I. TABLE I.

RO 0. 0.2i 0.4i 0.6i 0.8i 0.999i (RO is the main diagonal entry of Bc)

$$ev_{BC} 0. 5i 0.6i 0.7i 0.8i 0.9i 0.9995$$

$$ev_{DC} 0.8 + 0.4i, 0.735 + 0.4412i, 0.6098 + 0.4898i, 0.5525 + 0.4525i, 0.50050, 0.4999999i$$

Table I shows that,

$$ev_{BC} = 0.5i + RO / 2$$

and validate,

$$ev_{DC} = \text{Sum of power series: } ev_{Bc} + ev_{Bc}^2 + ev_{Bc}^3 + \dots + ev_{Bc}^N, \text{ for infinitely large } N.$$

In accordance with the results obtained for the case of real matrix B [3,4,5]

III-B SYMMETRIC AND SUPERSYMMETRIC MATRIX B

All B-stochastic transition matrices are symmetric but not all B matrices will be super symmetric.

The notion of super symmetric matrix B will be useful to describe the 4D hypercube by a super symmetric matrix B of 16 inputs and 16 outputs as illustrated in section III-C-

The super symmetric matrix B is a special case of the symmetric matrix B where more restrictions or conditions on the spatial geometry of the matrix B and therefore on the conditions i-iv are imposed.

In all cases,for super symmetric matrix ,call it Bs, the following conditions are satisfied:

- I-The sum of the entries $B_{i,j}$ for all the rows is equal to $1/2$,for super symmetric B matrix.
- II-The sum of the entries $B_{i,j}$ for all the columns is equal to $1/2$,for super symmetric B matrix.
- III-iiiThe sum of the entries $D_{i,j}$ for all the rows is equal to 1,for super symmetric D matrix.
- IV-The sum of the entries $D_{i,j}$ for all the columns is equal to 1,for super symmetric D matrix.

. The super symmetry of matrix B results in super symmetry of matrix E and its destination transfer matrix D .

Fig.1,2,3,4 illustrate symmetry vs super symmetry.

Fig.1 symmetric B-Martix 9x9 For 9 equidistant free nodes in 2D space.

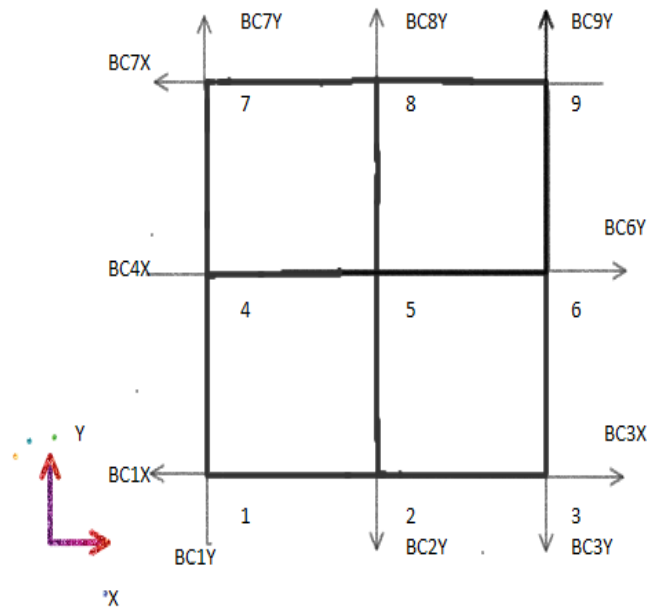


Fig.2 super- symmetric B-Martix 4x4 For 4 equidistant free nodes in 2D space.

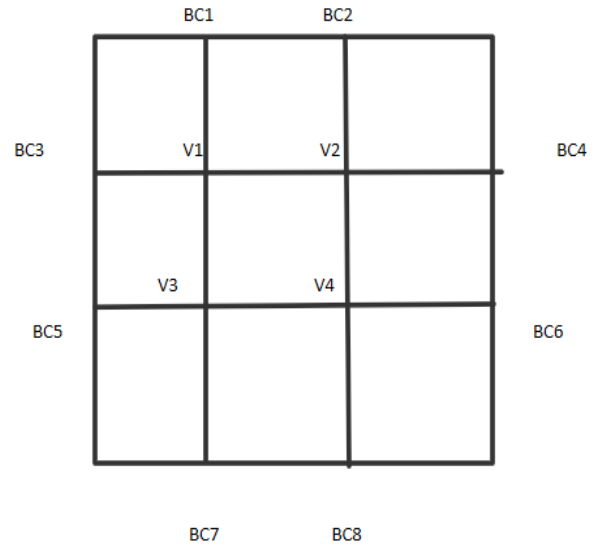


FIG.1

Fig.3 symmetric B-Martix 27x27 For 27 equidistant free nodes in 3D space.

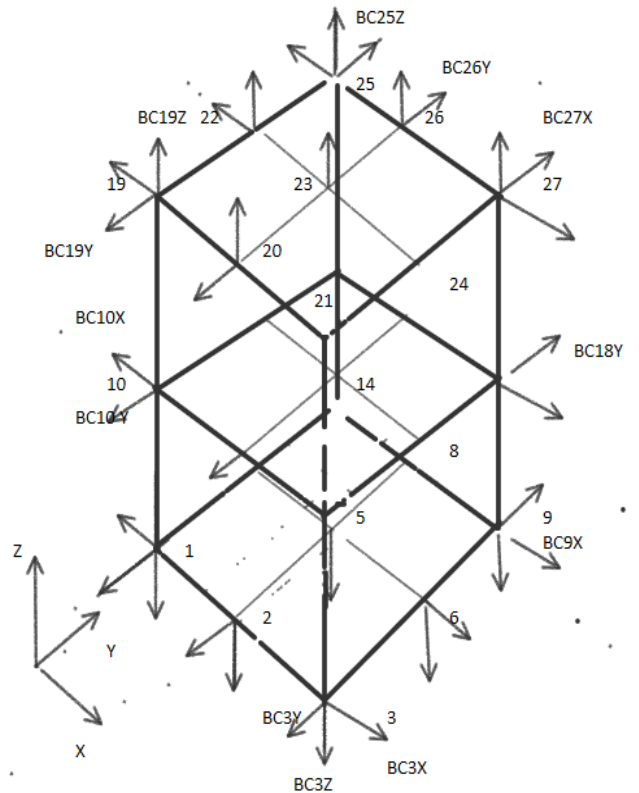


Fig.4 super-symmetric B-Martix 8x8 For 8 equidistant free nodes in 3D space.

