Bayesian Estimation of the Concentration Parameter of the L-Mode Von Mises Distribution

Randhir Singh
Department of Statistics, Ewing Christian College, Prayagraj, India

Abstract:- This paper deals with the Bayesian estimation of the concentration parameter κ of the l-mode Von Mises distribution under the assumption of four types of loss functions and a non-conjugate prior density for the unknown parameter. On the part of loss functions, the Squared Error Loss Function (SELF), DeGroot Loss Function (DLF), Minimum Expected Loss (MELO) Function and Exponentially Weighted Minimum Expected Loss (EWMELO) Function have been considered. Bayes Risks of Bayes estimators corresponding to four loss functions have also been obtained.

***Keywords**:- L-Mode Von Mises Distribution, Bayes Estimator, Squared Error Loss Function (SELF), Degroot Loss Function (DLF), Minimum Expected Loss (MELO) Function And Exponentially Weighted Minimum Expected Loss (EWMELO) Function. Bayes Risk.*

I. INTRODUCTION

Von Mises (1918), introduced a probability distribution in a study related with atomic weight. The distribution involved a circular random variable where observations are in the form of angles .A circular random Θ is said to have Von Mises distribution with concentration parameter κ and mean angle θ_0 if its probability density function ,denoted by f (θ/ κ) ,is as follows:

$$f(\theta/\kappa) = \begin{cases} \frac{\exp[\kappa\cos(\theta-\theta_0)}{2\pi I_0(\kappa)}, \text{if } -\pi < \theta < \pi, \kappa > 0 \\ 0, \text{Otherwise.} \end{cases} \quad (1.1)$$

κ is known as the concentration parameter.This distribution has unique mode at θ_0. $I_0(\kappa)$ is the modified Bessel function of first kind and order zero A generalized form of this distribution ,known as the l-mode Von Mises distribution is of the form as follows:

$$f(\theta/\kappa) = \begin{cases} \frac{\exp[\kappa\cos(l(\theta-\theta_0))]}{2\pi I_0(\kappa)}, \text{if } -\pi < \theta < \pi, \kappa > 0 \\ 0, \text{Otherwise.} \end{cases} \quad (1.2)$$

Where, l is a natural number. In this case the distribution has l-modes situated at $\frac{2\pi}{l}$ distant apart from the first mode at θ_0.The case ,when l=3 and $\theta_0 = -\frac{2\pi}{3}$ is of special importance as it gives gives the distribution of torsional moment of methanol molecule. Bagchi (1987) has worked on Bayesian analysis of directional data, Bagchi and Guttmann (1988) considered multivariate Von-Mises-Fisher distribution. Deely and Lindley (1981), Rodrigue et al (2000) have considered empirical Bayes method. Lindley and Smith(1972) considered the estimation with conjugated prior for the mean direction and a non-informative prior for the concentration parameter. The entropy of the distribution and the maximum likelihood estimation of the concentration parameter has been considered in the work of Demchuk and Singh (2001).

In this paper, Bayesian procedure has been adopted to obtain estimates of κ. The prior p. d. f of κ has been assumed to be of the form given as under :

$$g(\kappa) = \begin{cases} c\{I_0(\kappa)\}^n e^{-\kappa\lambda}\kappa^{\alpha+1}, \text{if } \kappa > 0, \lambda > 0, \alpha > 0 \\ 0, \text{Otherwise.} \end{cases} \quad (1.3)$$

Where, λ and α are known and c is the normalizing constant.

The loss functions considered are as under:
1.The Squared Error Loss Function (SELF): In this case, the loss function denoted by $L(\kappa,\delta)$,
is given by,
$L(\kappa,\delta) = (\kappa-\delta)^2$ (1.4)

This loss function is symmetric and unbounded. It suffers from the drawback of giving equal weights to underestimation as well as to overestimation.

2. DeGroot Loss Function (DLF): In this case
$L(\kappa,\delta) = \delta^{-2}(\kappa-\delta)^2$ (1.5)

This loss function, introduced by DeGroot (2005), is asymmetric. It gives more weight to underestimation than to overestimation.

3.Minimum Expected Loss (MELO) Function: In this case,
$L(\kappa,\delta) = \kappa^{-2}(\kappa-\delta)^2$ (1.6)

This loss function is asymmetric and bounded. In this case weight due to underestimation and overestimation is changed by a factor κ^{-2} as compared to the SELF. This loss function was used by Tummala and Sathe (1978) for estimating reliability of certain life time distribution and by Zellner (1979) for estimating functions of parameters in econometric models.

4. Exponentially Weighted Minimum Expected Loss (EWMELO) Function

$L(\kappa,\delta)=\kappa^{-2}e^{-a\kappa}(\kappa-\delta)^2$ (1.7)

Where, $a>0$ .This loss function is asymmetric and bounded. In this case weight due to underestimation and overestimation is changed by a factor $e^{-a\kappa}$ as compared to the MELO and by a factor $\kappa^{-2}e^{-a\kappa}$ as compared to the SELF.

This type of loss function was used by the author (1997) for the first time in his work for D.Phil. SELF, MELO and EWMELO were used by Singh, the author, (1999) in the study of reliability of a multicomponent system and (2010) in Bayesian Estimation of the mean and distribution function of Maxwell's distribution. Recently, the author again used these loss functions in Bayesian estimation of function of the unknown parameter θ for the Modified Power Series Distribution (MPSD) (2021) ,for estimating Loss and Risk Functions of a continuous distribution (2021),for estimating moments and reliability of Geometric distribution. In addition to these loss functions, the author has used Degroot loss function while estimating the unknown parameter and reliability of Burr Type XII distribution (2021) and Weibexpo distribution (2021).

II. BAYESIAN ESTIMATION

Let $\Theta_1,\Theta_2,\ldots.\Theta_n$ be a random sample of size n from the distribution specified by (1.2),where, θ_0 and l are known. Then, the statistic

$T_n=\sum_{i=1}^{n}\cos(l(\Theta_i-\theta_0))$ (2.1)

is sufficient statistic for κ.

For observed value θ_i of Θ_i (i=1,2…n), the likelihood function, denoted by $L(\kappa)$, is given by

$L(\kappa)=\left(\frac{1}{2\pi I_0(\kappa)}\right)^n e^{\kappa t_n}$ (2.2)

t_n is observed value of T_n.

The posterior p. d. f ,denoted by, $g_*(\kappa\,/\,t_n)$ is given by,

$g_*(\kappa\,/\,t_n)=\frac{L(\kappa)g(\kappa)}{\int_0^\infty L(\kappa)g(\kappa)d\kappa}$

Or,

$$g_*(\kappa\,/\,t_n)=\begin{cases}\frac{(\lambda-t_n)^{\alpha+2}\kappa^{\alpha+1}e^{-\kappa(\lambda-t_n)}}{\Gamma(\alpha+2)}, & \text{if } \kappa>0,\lambda-t_n>0\\ 0, & \text{Otherwise.}\end{cases} \quad (2.3)$$

It is to be noted that the posterior p. d. f. $g_*(\kappa\,/\,t_n)$,is the p. d. f. of the Gamma distribution with shape parameter $(\alpha+2)$ and scale parameter $\frac{1}{\lambda-t_n}$.Since, the prior does not correspond to the p. d. f. of the Gamma distribution, it is a non-natural conjugate prior distribution.

1.The Bayes estimate of κ ,under SELF, denoted by $\hat{\kappa}_B$,is given by,

$\hat{\kappa}_B=\int_0^\infty \kappa g_*(\kappa\,/\,t_n)d\kappa=\frac{\alpha+2}{\lambda-t_n}$ (2.4)

Provided ,$\lambda>t_n$

2.Under the DeGroot Loss Function, the Bayes estimate of κ , denoted by $\hat{\kappa}_{DG}$, is given by,

$\hat{\kappa}_{DG}=\frac{E(\kappa^2/t_n)}{E(\kappa/t_n)}=\frac{\int_0^\infty\kappa^2 g_*(\kappa\,/\,t_n)d\kappa}{\int_0^\infty\kappa g_*(\kappa\,/\,t_n)d\kappa}=\frac{\alpha+3}{\lambda-t_n}$ (2.5)

provided, $\lambda>t_n$

3.The Bayes estimate of κ ,under Minimum Expected Loss (MELO) ,denoted by $\hat{\kappa}_M$,is given by,

$\hat{\kappa}_M=\frac{\int_0^\infty\kappa^{-2}\kappa g(\kappa\,/\,t_n)d\kappa}{\int_0^\infty\kappa^{-2}g(\kappa\,/\,t_n)d\kappa}=\frac{\alpha}{\lambda-t_n}$ (2.6)

Provided ,$\lambda>t_n$

4.The Bayes estimate of κ ,under Exponentially Weighted Minimum Expected Loss (EWMELO) ,denoted by $\hat{\kappa}_E$,is given by,

$\hat{\kappa}_E=\frac{\int_0^\infty\kappa^{-2}e^{-a\kappa}\kappa g(\kappa\,/\,t)d\kappa}{\int_0^\infty\kappa^{-2}e^{-a\kappa}g(\kappa\,/\,t)d\kappa}=\frac{\alpha}{\lambda+a-t_n}$ (2.7)

Provided ,$\lambda+a>t_n$

The Bayes risk of a Bayes estimator $\hat{\kappa}$ of κ,corresponding to a given loss function $L(\kappa,\delta)$ is given by,$B(\hat{\kappa})=E\{L(\kappa,\hat{\kappa})\}$.Bayes risks of Bayes estimators corresponding to four loss functions considered are in the table as follows:

2.1BAYES RISKS OF VARIOUS BAYES ESTIMATES OF κ

S. No.	Loss Function	Bayes Estimate	Bayes Risk
1.	SELF	$\hat{\kappa}_B=\frac{\alpha+2}{\lambda-t_n}$	$B(\hat{\kappa}_B)=\frac{\alpha+2}{(\lambda-t_n)^2}$
2.	DLF	$\hat{\kappa}_{DG}=\frac{\alpha+3}{\lambda-t_n}$	$B(\hat{\kappa}_{DG})=\frac{1}{\alpha+3}$
3.	MELO	$\hat{\kappa}_M=\frac{\alpha}{\lambda-t_n}$	$B(\hat{\kappa}_M)=\frac{1}{\alpha+1}$
4.	EWMELO	$\hat{\kappa}_E=\frac{\alpha}{\lambda+a-t_n}$	$B(\hat{\kappa}_E)=\frac{1}{\alpha+1}\left(\frac{(\lambda-t_n)}{(\lambda+a-t_n)}\right)^{\alpha+2}$

It is to be noted that $B(\hat{\kappa}_E)<B(\hat{\kappa}_M)<B(\hat{\kappa}_{DG})$.

III. CONCLUSION

In this paper the Bayesian estimation of the concentration parameter κ of the l-mode Von Mises Distribution has been obtained under the assumption of a non-conjugate prior density for the unknown parameter and corresponding to four different types of loss functions. Bayes Risks of Bayes estimators corresponding to four loss functions have also been obtained. It has been observed that among three loss functions, namely DLF, MELO and EWMELO respectively, EWMELO has the minimum Bayes Risk.

REFERENCES

[1]. Bagchi, P (1987): Bayesian Analysis of Directional Data ,Ph.D.Thesis University of Toranto.
[2]. Bagchi, P & Guttman, I (1988):Theoretical Considerations of the Multinomial VonMises –Fisher Distribution, Journal of Applied Statistics ,15,149 -169.
[3]. Deely, J.J. & Lindley ,D.(1981) Bayes Empirical Bayes,JASA,76,833-841
[4]. DeGroot, M.H. (2005), Optimal Statistical Decisions, Vol. 82, John Wiley & Sons
[5]. Demchuk, Eugene and Singh, Harshinder (2001):Statistical thermodynamics of hindered rotation from computer simulations, Molecular Physics,Vol.99,N0.8,pp.627-636
[6]. Lindley ,D.V.& Smith,A.F.M. (1972)Bayes Estimate for the Linear Model (with discussions);J. R .Stat. Soc. Series B, Statistical Methodology,34,1-41
[7]. Josemar Rodrigue ,Jos$\dot{e}$Galv$\bar{a}$o Liete &Luis ,A,Milan (2000): Empirical Bayes Inference for the Von Mises distribution. Aust. N.Z.Stat 42 (4),433-440
[8]. Singh, Randhir. (1997), D.Phil Thesis (Unpublished), Department of Mathematics and Statistics, University of Allahabad, Allahabad (INDIA).
[9]. Singh, Randhir .(1999), "Bayesian Analysis of a Multicomponent System" ,Proceedings of NSBA-TA, 16-18 Jan.1999,pp.252-261.Editor -Dr. Rajesh Singh. The conference was organised by the Department of Statistics, Amrawati University, Amrawati-444602. Maharastra,India.
[10]. Singh, Randhir .(2010), "Simulation Aided Bayesian Estimation for Maxwell's Distribution", Proceedings of National Seminar on Impact of Physics on Biological Sciences (August,26,2010),held by the Department of Physics, Ewing Christian College, Prayagraj,India,pp.203-210; ISBN No.: 978-81-905712-9-6
[11]. Singh Randhir .(2021), "On Bayesian Estimation of Loss and Risk Functions. Science Journal of Applied Mathematics and Statistics", Vol.9, No.3,2021, pp.73-77.doi: 10.11648/j.sjams.20210903.11
[12]. Singh, Randhir. (2021):"On Bayesian Estimation of Function of Unknown Parameter of Modified Power Series Distribution". International Journal of Innovative Science and Research Technology ISSN No:2456-2165, Volume 6,Issue 6, June-2021,pp.861-864 www.ijirst.com
[13]. Singh, Randhir. (2021):" Bayesian Estimation of Function of Unknown Parameters of Some Particular Cases of Modified Power Series Distribution". Journal of Emerging Technologies and Innovative Research (JETIR), ISSN No:2349-5162, Volume 8,Issue 7, pp.d673-d681 www.jetir.org
[14]. Singh, Randhir. (2021): Bayesian Estimation of Moments and Reliability of Geometric Distributtion, Quest Journals : Journal of Research in Applied Mathematics,Volume7~Issue 7(2021) pp:19-25. www.questjournals.org
[15]. Singh, Randhir. (2021): Bayesian and Classical Estimation of Parameter and Reliability of Burr Type XII Distribution. Journal of Emerging Technologies and Innovative Research (JETIR), ISSN No:2349-5162, Volume 8,Issue 9, pp.a546-a556 www.jetir.org
[16]. Singh, Randhir. (2021):" Bayesian and Classical Estimation of Parameter and Reliability of Weibexpo Distribution". International Journal of Innovations in Engineering Research and Technology and Innovative Research (IJIERT), ISSN(E):2394-3696, Volume 8,Issue 9, pp.131 -139 www.ijiert.org
[17]. Tummala ,V.M.& Sathe ,P.T. (1978) MELO Estimators of Reliability and Parameters of certain Life time distributions ,IEEE Transactions on Reliability,Vol.R-27,No .4 pp 283-285
[18]. Von Mises, R.(1918) Physikal .Z.,19,490
[19]. Zellner ,A. & Park ,S.B. (1979) Minimum Expected Loss (MELO) Estimators of functions of parameters and structural coefficients of econometric models ; JASA ,74,pp185-193.